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# The Differential Invariants of Space

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# VIII. *The Differential Invariants of Space.*

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The present memoir is intended to carry out investigations, concerned with the differential invariants of ordinary space and of surfaces in that space, similar to those in a former memoir,* concerned with the differential invariants of a surface and of curves upon that surface.

The method used in the former memoir is used here in what is the obviously natural development. It is based upon the method,† which was originated by Lie and amplified by Professor Žorawski. When applied to two-dimensional invariants, it proved possible to modify and simplify the later stages of the calculations by making them dependent upon the concomitants of a system of simultaneous binary forms. When applied to three-dimensional invariants, it proves possible to effect a corresponding simplification in the later stages of the calculations; the required functions are found to be the invariants and the contravariants of a system of simultaneous ternary forms.

The expressions for an algebraically complete aggregate of invariants up to the third order inclusive have been obtained. The calculations necessary for the construction of these invariants were laborious; indeed, the calculations for the invariants of the third order are so long that in this memoir they have been suppressed, and only the results are given. It may be mentioned incidentally that, among the invariants of the third order, six (in particular) occur possessing a special property. They can be so taken in the algebraically complete aggregate as to coincide with six quantities which were proved by Cayley to vanish on account of the intrinsic significance of the fundamental magnitudes. These six equations are the generalisation, to surfaces not orthogonal, of Lamé's six equations for triply orthogonal surfaces.

The geometric significance of practically all the differential invariants of the first order and the second order has been obtained. I have not yet attempted to identify

* 'Phil. Trans.,' A, vol. 201 (1903), pp. 329–402.

† References are given in the memoir just quoted.

the invariants proper to the third order; there is reason to suppose that, when they are completely identified in association with even a single surface in space, they can be used to establish two fundamental geometrical relations affecting the surface.

## *The Fundamental Magnitudes.*

1. The independent variables of position in space are taken to be $u$, $v$, $w$. That position is also defined in the customary manner by rectangular Cartesian co-ordinates $x$, $y$, $z$; and then $u$, $v$, $w$ may be regarded as three independent functions of $x$, $y$, $z$. Conversely, we shall assume that $x$, $y$, $z$ are expressible as functions of $u$, $v$, $w$, which are regular in the vicinity of any assigned position, save for singular lines or points with which we are not concerned. Moreover, if

$$u = u(x, y, z), \quad v = v(x, y, z), \quad w = w(x, y, z),$$

the surfaces $u$ = constant, $v$ = constant, $w$ = constant, form a triple family; it will not be assumed that the triple family is orthogonal. The parameters $u$, $v$, $w$ of the surfaces are sometimes called curvilinear co-ordinates.

Fundamental magnitudes for space arise in the expression of a distance-element in terms of $u$, $v$, $w$, $du$, $dv$, $dw$. Denoting this element by $ds$, we have

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= a\,du^2 + 2h\,du\,dv + 2g\,du\,dw + b\,dv^2 + 2f\,dv\,dw + c\,dw^2 \\ &= (a, b, c, f, g, h \between du, dv, dw)^2, \end{aligned}$$

in the usual notation, where

$$a = \Sigma\left(\frac{\partial x}{\partial u}\right)^2, \qquad b = \Sigma\left(\frac{\partial x}{\partial v}\right)^2, \qquad c = \Sigma\left(\frac{\partial x}{\partial w}\right)^2,$$

$$f = \Sigma\,\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}, \qquad g = \Sigma\,\frac{\partial x}{\partial w}\frac{\partial x}{\partial u}, \qquad h = \Sigma\,\frac{\partial x}{\partial u}\frac{\partial x}{\partial v},$$

the summation being taken over the three variables $x$, $y$, $z$ in each case.

The quantities $a$, $b$, $c$, $f$, $g$, $h$ may be called the fundamental magnitudes of the first kind, as involving derivatives of only the first order. Other fundamental magnitudes may exist in association with derivatives of higher orders;* they are ignored for the purposes of the present investigation.

* Those magnitudes, which are the natural generalisation of the magnitudes L, M, N in the Gaussian theory of surfaces, are not independent quantities; they are proved by CAYLEY, 'Coll. Math. Papers,' vol. 12, p. 4, to be expressible in terms of derivatives of $a$, $b$, $c$, $f$, $g$, $h$.

It may be added that the memoir by CAYLEY, which has just been quoted, contains the establishment of the six intrinsic equations mentioned in the introductory remarks. For reasons which will appear in the course of the memoir, I have found it desirable to deviate to some extent from CAYLEY'S notation.

*Property of Invariance.*

2. A combination F of any number of functions of $u$, $v$, $w$ and of the derivatives of these functions is said to be a relative invariant if, when any new independent variables $u'$, $v'$, $w'$ are introduced, and the same combination F′ of the modified functions and of their derivatives is formed, the relation

$$F = \Omega^{\mu} F'$$

is satisfied, where

$$\Omega = \frac{\partial (u', v', w')}{\partial (u, v, w)}.$$

The invariants actually considered are rational, so that $\mu$ is an integer. The invariant is called absolute when $\mu = 0$.

It is a known theorem that the property of invariance is possessed in general, if it is possessed for the most general infinitesimal transformation; we shall therefore take

$$u' = u + \xi (u, v, w)\, dt,$$

$$v' = v + \eta (u, v, w)\, dt,$$

$$w' = w + \zeta (u, v, w)\, dt,$$

where $\xi$, $\eta$, $\zeta$ are arbitrary integral functions of $u$, $v$, $w$, and $dt$ is an infinitesimal quantity of which only the first power is retained.

Derivatives with regard to $u$, $v$, $w$ are required; we write

$$\theta_{lmn} = \frac{\partial \theta^{l+m+n}}{\partial u^l\, \partial v^m\, \partial w^n}$$

for any quantity $\theta$ and for all values of $l$, $m$, $n$. With this notation, we at once have the retained value of $\Omega$ in the form

$$\Omega = 1 + (\xi_{100} + \eta_{010} + \zeta_{001})\, dt.$$

*Arguments of the Invariants and their Increments.*

3. As regards the possible arguments of a differential invariant of space, we shall have the fundamental magnitudes of the first kind and their derivatives. It is conceivable that properties of surfaces in the space and of curves in the space will be involved; provision for the possibility will be made by the introduction of functions such as $\phi (u, v, w)$. One of these, equated to zero or to a constant, will give a surface: two such surfaces will give a curve or curves: three such surfaces will give a point or points.

It is not difficult to see that $u$, $v$, $w$ will not occur explicitly in the expression of the invariant, nor will any function of them such as $\phi (u, v, w)$ occur explicitly; but derivatives of such a function as $\phi$ will occur, and the order of the derivatives will depend upon the order of the derivatives of $a, b, c, f, g, h$ which arise in the invariant.

Clearly one invariant is provided by the determinant

$$L^2 = \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix};$$

for on substituting the values of the fundamental magnitudes in terms of the derivatives of $x, y, z$, we have

$$\begin{aligned} L &= \frac{\partial(x, y, z)}{\partial(u, v, w)} \\ &= \Omega L', \end{aligned}$$

so that L is an invariant for which the index $\mu$ is 1.

It is convenient to denote the minors of the constituents in $L^2$ by A, B, C, F, G, H, where

$$A = bc - f^2, \qquad B = ca - g^2, \qquad C = ab - h^2,$$

$$F = gh - af, \qquad G = hf - bg, \qquad H = fg - ch.$$

4. The laws of transformation of the various magnitudes are required in order to construct their increments which correspond to the increments of $u, v, w$. Writing

$$\begin{aligned} ds^2 &= (a, b, c, f, g, h \between du, dv, dw)^2 \\ &= (a', b', c', f', g', h' \between du', dv', dw')^2, \end{aligned}$$

we have

$$\begin{aligned} a &= \left(a', b', c', f', g', h' \between \frac{\partial u'}{\partial u}, \frac{\partial v'}{\partial u}, \frac{\partial w'}{\partial u}\right)^2 \\ &= a'(1 + 2\xi_{100}\, dt) + 2h'\eta_{100}\, dt + 2g'\zeta_{100}\, dt \\ &= a' + (2a\xi_{100} + 2h\eta_{100} + 2g\zeta_{100})\, dt, \end{aligned}$$

on neglecting squares and higher powers of $dt$. Similarly

$$\begin{aligned} b &= b' + (2h\xi_{010} + 2b\eta_{010} + 2f\zeta_{010})\, dt, \\ c &= c' + (2g\xi_{001} + 2f\eta_{001} + 2c\zeta_{001})\, dt, \\ f &= f' + \{h\xi_{001} + g\xi_{010} + b\eta_{001} + f(\eta_{010} + \zeta_{001}) + c\zeta_{010}\}\, dt, \\ g &= g' + \{a\xi_{001} + h\eta_{001} + g(\xi_{100} + \zeta_{001}) + f\eta_{100} + c\zeta_{100}\}\, dt, \\ h &= h' + \{a\xi_{010} + h(\xi_{100} + \eta_{010}) + g\zeta_{010} + b\eta_{100} + f\zeta_{100}\}\, dt. \end{aligned}$$

These six equations at once give the increments of $a, b, c, f, g, h$, They can also be used to construct the increments of the derivatives of fundamental magnitudes, as follows.

5. Let $u$, $v$, $w$ become $u+i$, $v+j$, $w+k$; and let the consequent values of $u'$, $v'$, $w'$ be $u'+i'$, $v'+j'$, $w'+k'$; then

$$i' = i + \{\xi(u+i, v+j, w+k) - \xi(u, v, w)\}\,dt$$
$$= i + \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\xi_{lmn}\,dt,$$
$$j' = j + \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\eta_{lmn}\,dt,$$
$$k' = k + \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\zeta_{lmn}\,dt,$$

where $\Sigma\Sigma\Sigma'$ implies summation for all positive integer (including zero) values of $l$, $m$, $n$, save only simultaneous zero values. Hence also for all integer values of $p$, $q$, $r$, we have

$$i'^p j'^q k'^r = i^p j^q k^r + p i^{p-1} j^q k^r \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\xi_{lmn}\,dt$$
$$+ q i^p j^{q-1} k^r \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\eta_{lmn}\,dt$$
$$+ r i^p j^q k^{r-1} \Sigma\Sigma\Sigma' \frac{i^l j^m k^n}{l!\,m!\,n!}\zeta_{lmn}\,dt.$$

Now

$$a(u+i, v+j, w+k) = a(u'+i', v'+j', w'+k')$$
$$+ 2a(u+i, v+j, w+k)\,\xi_{100}(u+i, v+j, w+k)\,dt$$
$$+ 2h(u+i, v+j, w+k)\,\eta_{100}(u+i, v+j, w+k)\,dt$$
$$+ 2g(u+i, v+j, w+k)\,\zeta_{100}(u+i, v+j, w+k)\,dt.$$

Expand the first term on the right-hand side in powers of $i'$, $j'$, $k'$, and substitute for all combinations such as $i'^p j'^q h'^r$ in terms of $i$, $j$, $k$; expand also all the other quantities in powers of $i$, $j$, $k$, and select the coefficient of $i^\alpha j^\beta k^\gamma$. Let

$$\binom{s}{t} = \frac{s!}{(s-t)!\,t!},$$

$$a'_{\alpha\beta\gamma} = a_{\alpha\beta\gamma} + \frac{da_{\alpha\beta\gamma}}{dt}\,dt;$$

then, neglecting powers of $dt$ higher than the first and multiplying up by $\alpha!\,\beta!\,\gamma!$, we have

$$-\frac{da_{\alpha\beta\gamma}}{dt} = \Sigma\Sigma\Sigma'\binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\{a_{\alpha+1-l,\,\beta-m,\,\gamma-n}\xi_{lmn} + a_{\alpha-l,\,\beta+1-m,\,\gamma-n}\eta_{lmn} + a_{\alpha-l,\,\beta-m,\,\gamma+1-n}\zeta_{lmn}\}$$
$$+ 2\Sigma\Sigma\Sigma\binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\{a_{\alpha-l,\,\beta-m,\,\gamma-n}\xi_{l+1,\,m,\,n} + h_{\alpha-l,\,\beta-m,\,\gamma-n}\eta_{l+1,\,m,\,n} + g_{\alpha-l,\,\beta-m,\,\gamma-n}\zeta_{l+1,\,m,\,n}\},$$

where the summations are for all integer values of $l$ from 0 to $\alpha$, of $m$ from 0 to $\beta$, and of $n$ from 0 to $\gamma$, the simultaneous zero values being excluded from the triple summations $\Sigma\Sigma\Sigma'$ in the first line of the right-hand side.

Proceeding in the same way with the expression for the increment of $f$, we find

$$-\frac{df_{\alpha\beta\gamma}}{dt} = \Sigma\Sigma\Sigma'\binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\{f_{\alpha+1-l,\,\beta-m,\,\gamma-n}\xi_{lmn} + f_{\alpha-l,\,\beta+1-m,\,\gamma-n}\eta_{lmn} + f_{\alpha-l,\,\beta-m,\,\gamma+1-n}\zeta_{lmn}\}$$

$$+ \Sigma\Sigma\Sigma\binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\{h_{\alpha-l,\,\beta-m,\,\gamma-n}\xi_{l,\,m,\,n+1} + g_{\alpha-l,\,\beta-m,\,\gamma-n}\xi_{l,\,m+1,\,n}$$

$$+ b_{\alpha-l,\,\beta-m,\,\gamma-n}\eta_{l,\,m,\,n+1} + f_{\alpha-l,\,\beta-m,\,\gamma-n}\eta_{l,\,m+1,\,n}$$

$$+ f_{\alpha-l,\,\beta-m,\,\gamma-n}\zeta_{l,\,m,\,n+1} + c_{\alpha-l,\,\beta-m,\,\gamma-n}\zeta_{l,\,m+1,\,n}\},$$

where again the summations are for all integer values of $l$ from 0 to $\alpha$, of $m$ from 0 to $\beta$, and of $n$ from 0 to $\gamma$, the simultaneous zero values being excluded from the triple summations $\Sigma\Sigma\Sigma'$ in the first line of the right-hand side.

Effecting in these two results all the interchanges that correspond to the interchange of the variables $u$ and $v$, we obtain the values of

$$-\frac{db_{\alpha\beta\gamma}}{dt}, \qquad -\frac{dg_{\alpha\beta\gamma}}{dt};$$

and also effecting in them all the interchanges that correspond to the interchange of the variables $u$ and $w$, we obtain the values of

$$-\frac{dc_{\alpha\beta\gamma}}{dt}, \qquad -\frac{dh_{\alpha\beta\gamma}}{dt}.$$

The expression for the increment of the derivatives of any function $\phi(u, v, w)$, where $\phi$ is unaltered in value by transformation, can be obtained in the same way as was that for the increment of the derivatives of $a$; it is found to be

$$-\frac{d\phi_{\alpha\beta\gamma}}{dt} = \Sigma\Sigma\Sigma'\binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\{\phi_{\alpha+1-l,\,\beta-m,\,\gamma-n}\xi_{lmn} + \phi_{\alpha-l,\,\beta+1-m,\,\gamma-n}\eta_{lmn} + \phi_{\alpha-l,\,\beta-m,\,\gamma+1-n}\zeta_{lmn}\},$$

with the same significance for $\Sigma\Sigma\Sigma'$ as before.

It thus appears that if the highest order of derivatives of the fundamental magnitudes that occur in a differential invariant be M, the highest order of derivatives of a function $\phi$ that can occur is M + 1.

6. In order to avoid encumbering the memoir with vast masses of symbols, I propose to exhibit the mode of constructing differential invariants up to the second order, that is, invariants involving derivatives of a single function $\phi$ up to the second order and derivatives of $a, b, c, f, g, h$ of the first order. Then I propose to indicate what are the differential invariants up to the second order that involve more than a single function $\phi$. And as the last part of the merely analytical portion of the memoir, I propose to state the results for differential invariants up to the third order but to give practically none of the contributory analysis. Some idea of the protracted

character of the calculations needed for the construction of the differential invariants of the third order may be gathered from the fact, that they require the complete solution of a Jacobian system of fifty-seven simultaneous partial differential equations of the first order and the first degree.

For such analysis as will here be given in detail, we need the following particular cases of the preceding general results :—

$$-\frac{da}{dt} = 2a\xi_{100} + 2h\eta_{100} + 2g\zeta_{100},$$

$$-\frac{db}{dt} = 2h\xi_{010} + 2b\eta_{010} + 2f\zeta_{010},$$

$$-\frac{dc}{dt} = 2g\xi_{001} + 2f\eta_{001} + 2c\zeta_{001},$$

$$-\frac{df}{dt} = g\xi_{010} + h\xi_{001} + f\eta_{010} + b\eta_{001} + c\zeta_{010} + f\zeta_{001},$$

$$-\frac{dg}{dt} = g\xi_{100} + a\xi_{001} + f\eta_{100} + h\eta_{001} + c\zeta_{100} + g\zeta_{001},$$

$$-\frac{dh}{dt} = h\xi_{100} + a\xi_{010} + b\eta_{100} + h\eta_{010} + f\zeta_{100} + g\zeta_{010};$$

$$-\frac{da_{100}}{dt} = 3a_{100}\xi_{100} + 2a\xi_{200} + (a_{010} + 2h_{100})\,\eta_{100} + 2h\eta_{200} + (a_{001} + 2g_{100})\,\zeta_{100} + 2g\zeta_{200},$$

$$-\frac{da_{010}}{dt} = 2a_{010}\xi_{100} + a_{100}\xi_{010} + 2a\xi_{110} + 2h_{010}\eta_{100} + a_{010}\eta_{010} + 2h\eta_{110} + 2g_{010}\zeta_{100} + a_{001}\zeta_{010} + 2g\zeta_{110},$$

$$-\frac{da_{001}}{dt} = 2a_{001}\xi_{100} + a_{100}\xi_{001} + 2a\xi_{101} + 2h_{001}\eta_{100} + a_{010}\eta_{001} + 2h\eta_{101} + 2g_{001}\zeta_{100} + a_{001}\zeta_{001} + 2g\zeta_{101};$$

$$-\frac{db_{100}}{dt} = b_{100}\xi_{100} + 2h_{100}\xi_{010} + 2h\xi_{110} + b_{010}\eta_{100} + 2b_{100}\eta_{010} + 2b\eta_{110} + b_{001}\zeta_{100} + 2f_{100}\zeta_{010} + 2f\zeta_{110},$$

$$-\frac{db_{010}}{dt} = (b_{100} + 2h_{010})\,\xi_{010} + 2h\xi_{020} + 3b_{010}\eta_{010} + 2b\eta_{020} + (b_{001} + 2f_{010})\,\zeta_{010} + 2f\zeta_{020},$$

$$-\frac{db_{001}}{dt} = 2h_{001}\xi_{010} + b_{100}\xi_{001} + 2h\xi_{011} + 2b_{001}\eta_{010} + b_{010}\eta_{001} + 2b\eta_{011} + 2f_{001}\zeta_{010} + b_{001}\zeta_{001} + 2f\zeta_{011};$$

$$-\frac{dc_{100}}{dt} = c_{100}\xi_{100} + 2g_{100}\xi_{001} + 2g\xi_{101} + c_{010}\eta_{100} + 2f_{100}\eta_{001} + 2f\eta_{101} + c_{001}\zeta_{100} + 2c_{100}\zeta_{001} + 2c\zeta_{101},$$

$$-\frac{dc_{010}}{dt} = c_{100}\xi_{010} + 2g_{010}\xi_{001} + 2g\xi_{011} + c_{010}\eta_{010} + 2f_{010}\eta_{001} + 2f\eta_{011} + c_{001}\zeta_{010} + 2c_{010}\zeta_{001} + 2c\zeta_{011},$$

$$-\frac{dc_{001}}{dt} = (c_{100} + 2g_{001})\,\xi_{001} + 2g\xi_{002} + (c_{010} + 2f_{001})\,\eta_{001} + 2f\eta_{002} + 3c_{001}\zeta_{001} + 2c\zeta_{002};$$

$$-\frac{df_{100}}{dt} = f_{100}\xi_{100} + g_{100}\xi_{010} + h_{100}\xi_{001} + g\xi_{110} + h\xi_{101}$$
$$+ f_{010}\eta_{100} + f_{100}\eta_{010} + b_{100}\eta_{001} + f\eta_{110} + b\eta_{101}$$
$$+ f_{001}\zeta_{100} + c_{100}\zeta_{010} + f_{100}\zeta_{001} + c\zeta_{110} + f\zeta_{101},$$

$$-\frac{df_{010}}{dt} = (f_{100} + g_{010})\,\xi_{010} + h_{010}\xi_{001} + g\xi_{020} + h\xi_{011}$$
$$+ 2f_{010}\eta_{010} + b_{010}\eta_{001} + f\eta_{020} + b\eta_{011}$$
$$+ (f_{001} + c_{010})\,\zeta_{010} + f_{010}\zeta_{001} + c\zeta_{020} + f\zeta_{011},$$

$$-\frac{df_{001}}{dt} = g_{001}\xi_{010} + (f_{100} + h_{001})\,\xi_{001} + g\xi_{011} + h\xi_{002}$$
$$+ f_{001}\eta_{010}\,(f_{010} + b_{001})\,\eta_{001} + f\eta_{011} + b\eta_{002}$$
$$+ c_{001}\zeta_{010} + 2f_{001}\zeta_{001} + c\zeta_{011} + f\zeta_{002}\,;$$

$$-\frac{dg_{100}}{dt} = 2g_{100}\xi_{100} + a_{100}\xi_{001} + g\xi_{200} + a\xi_{101}$$
$$+ (g_{010} + f_{100})\,\eta_{100} + h_{100}\eta_{001} + f\eta_{200} + h\eta_{101}$$
$$+ (g_{001} + c_{100})\,\zeta_{100} + g_{100}\zeta_{001} + c\zeta_{200} + g\zeta_{101},$$

$$-\frac{dg_{010}}{dt} = g_{010}\xi_{100} + g_{100}\xi_{010} + a_{010}\xi_{001} + g\xi_{110} + a\xi_{011}$$
$$+ f_{010}\eta_{100} + g_{010}\eta_{010} + h_{010}\eta_{001} + f\eta_{110} + h\eta_{011}$$
$$+ c_{010}\zeta_{100} + g_{001}\zeta_{010} + g_{010}\zeta_{001} + c\zeta_{110} + g\zeta_{011},$$

$$-\frac{dg_{001}}{dt} = g_{001}\xi_{100} + (a_{001} + g_{100})\,\xi_{001} + g\xi_{101} + a\xi_{002}$$
$$+ f_{001}\eta_{100} + (h_{001} + g_{010})\,\eta_{001} + f\eta_{101} + h_{002}$$
$$+ c_{001}\zeta_{100} + 2g_{001}\zeta_{001} + c\zeta_{101} + g\zeta_{002}\,;$$

$$-\frac{dh_{100}}{dt} = 2h_{100}\xi_{100} + a_{100}\xi_{010} + h\xi_{200} + a\xi_{110}$$
$$+ (h_{010} + b_{100})\,\eta_{100} + h_{100}\eta_{010} + b\eta_{200} + h\eta_{110}$$
$$+ (h_{001} + f_{100})\,\zeta_{100} + g_{100}\zeta_{010} + f\zeta_{200} + g\zeta_{110},$$

$$-\frac{dh_{010}}{dt} = h_{010}\xi_{100} + (a_{010} + h_{100})\,\xi_{010} + h\xi_{110} + a\xi_{020}$$
$$+ b_{010}\eta_{100} + 2h_{010}\eta_{010} + b\eta_{110} + h\eta_{020}$$
$$+ f_{010}\zeta_{100} + (g_{010} + h_{001})\,\zeta_{010} + f\zeta_{110} + g\zeta_{020},$$

$$-\frac{dh_{001}}{dt} = h_{001}\xi_{100} + a_{001}\xi_{010} + h_{100}\xi_{001} + h\xi_{101} + a\xi_{011}$$
$$+ b_{001}\eta_{100} + h_{001}\eta_{010} + h_{010}\eta_{001} + b\eta_{101} + h\eta_{011}$$
$$+ f_{001}\zeta_{100} + g_{001}\zeta_{010} + h_{001}\zeta_{001} + f\zeta_{101} + g\zeta_{011}\,;$$

$$-\frac{d\phi_{100}}{dt} = \phi_{100}\xi_{100} + \phi_{010}\eta_{100} + \phi_{001}\zeta_{100},$$

$$-\frac{d\phi_{010}}{dt} = \phi_{100}\xi_{010} + \phi_{010}\eta_{010} + \phi_{001}\zeta_{010},$$

$$-\frac{d\phi_{001}}{dt} = \phi_{100}\xi_{001} + \phi_{010}\eta_{001} + \phi_{001}\zeta_{001};$$

$$-\frac{d\phi_{200}}{dt} = 2\phi_{200}\xi_{100} + \phi_{100}\xi_{200} + 2\phi_{100}\eta_{100} + \phi_{010}\eta_{200} + 2\phi_{101}\zeta_{100} + \phi_{001}\zeta_{200},$$

$$-\frac{d\phi_{020}}{dt} = 2\phi_{110}\xi_{010} + \phi_{100}\xi_{020} + 2\phi_{020}\eta_{010} + \phi_{010}\eta_{020} + 2\phi_{011}\zeta_{010} + \phi_{001}\zeta_{020},$$

$$-\frac{d\phi_{002}}{dt} = 2\phi_{101}\xi_{001} + \phi_{100}\xi_{002} + 2\phi_{011}\eta_{001} + \phi_{010}\eta_{002} + 2\phi_{002}\zeta_{001} + \phi_{001}\zeta_{002},$$

$$-\frac{d\phi_{110}}{dt} = \phi_{110}\xi_{100} + \phi_{200}\xi_{010} + \phi_{100}\xi_{110} + \phi_{020}\eta_{100} + \phi_{110}\eta_{010} + \phi_{010}\eta_{110} + \phi_{011}\zeta_{100} + \phi_{101}\zeta_{010} + \phi_{001}\zeta_{110},$$

$$-\frac{d\phi_{101}}{dt} = \phi_{101}\xi_{100} + \phi_{200}\xi_{001} + \phi_{100}\xi_{101} + \phi_{011}\eta_{100} + \phi_{110}\eta_{001} + \phi_{010}\eta_{101} + \phi_{002}\zeta_{100} + \phi_{101}\zeta_{001} + \phi_{001}\zeta_{101},$$

$$-\frac{d\phi_{011}}{dt} = \phi_{101}\xi_{010} + \phi_{110}\xi_{001} + \phi_{100}\xi_{011} + \phi_{011}\eta_{010} + \phi_{020}\eta_{001} + \phi_{010}\eta_{011} + \phi_{002}\zeta_{010} + \phi_{011}\zeta_{001} + \phi_{001}\zeta_{011}.$$

*The Differential Equations characteristic of the Invariance.*

7. Let $\sigma$ denote any differential invariant involving at least some of the quantities whose increments due to an infinitesimal variation have just been given. The differential equations characteristic of the invariance can be deduced from the equation

$$\sigma = \Omega^{\mu}\sigma'$$

in the usual manner: we substitute for each argument $\theta'$ in $\sigma'$ its value

$$\theta + \frac{d\theta}{dt}\, dt,$$

and equate to zero the composite coefficient of $dt$ on the right-hand side. The quantities $\xi$, $\eta$, $\zeta$ are arbitrary and independent; the coefficients of the various derivatives in the last equation must therefore vanish. These relations are the partial differential equations which, up to the order of differentiation retained, are characteristic of the invariants; they are as follows:

There are the three equations

$$\left.\begin{aligned} 2a\frac{\partial\sigma}{\partial a_{100}}+h\frac{\partial\sigma}{\partial h_{100}}+g\frac{\partial\sigma}{\partial g_{100}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{200}}=0\\ 2h\frac{\partial\sigma}{\partial a_{100}}+b\frac{\partial\sigma}{\partial h_{100}}+f\frac{\partial\sigma}{\partial g_{100}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{200}}=0\\ 2g\frac{\partial\sigma}{\partial a_{100}}+f\frac{\partial\sigma}{\partial h_{100}}+c\frac{\partial\sigma}{\partial g_{100}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{200}}=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{II}_1),$$

from the coefficients of $\xi_{200}$, $\eta_{200}$, $\zeta_{200}$ respectively; the three equations

$$\left.\begin{aligned} a\left(2\frac{\partial\sigma}{\partial a_{010}}+\frac{\partial\sigma}{\partial h_{100}}\right)+h\left(2\frac{\partial\sigma}{\partial b_{100}}+\frac{\partial\sigma}{\partial h_{010}}\right)+g\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial g_{010}}\right)+\phi_{100}\frac{\partial\sigma}{\partial\phi_{110}}=0\\ h\left(2\frac{\partial\sigma}{\partial a_{010}}+\frac{\partial\sigma}{\partial h_{100}}\right)+b\left(2\frac{\partial\sigma}{\partial b_{100}}+\frac{\partial\sigma}{\partial h_{010}}\right)+f\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial g_{010}}\right)+\phi_{010}\frac{\partial\sigma}{\partial\phi_{110}}=0\\ g\left(2\frac{\partial\sigma}{\partial a_{010}}+\frac{\partial\sigma}{\partial h_{100}}\right)+f\left(2\frac{\partial\sigma}{\partial b_{100}}+\frac{\partial\sigma}{\partial h_{010}}\right)+c\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial g_{010}}\right)+\phi_{001}\frac{\partial\sigma}{\partial\phi_{110}}=0 \end{aligned}\right\} \quad . \quad (\mathrm{II}_2),$$

from the coefficients of $\xi_{110}$, $\eta_{110}$, $\zeta_{110}$ respectively; the three equations

$$\left.\begin{aligned} a\left(2\frac{\partial\sigma}{\partial a_{001}}+\frac{\partial\sigma}{\partial g_{100}}\right)+h\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial h_{001}}\right)+g\left(\frac{\partial\sigma}{\partial g_{001}}+2\frac{\partial\sigma}{\partial c_{100}}\right)+\phi_{100}\frac{\partial\sigma}{\partial\phi_{101}}=0\\ h\left(2\frac{\partial\sigma}{\partial a_{001}}+\frac{\partial\sigma}{\partial g_{100}}\right)+b\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial h_{001}}\right)+f\left(\frac{\partial\sigma}{\partial g_{001}}+2\frac{\partial\sigma}{\partial c_{100}}\right)+\phi_{010}\frac{\partial\sigma}{\partial\phi_{101}}=0\\ g\left(2\frac{\partial\sigma}{\partial a_{001}}+\frac{\partial\sigma}{\partial g_{100}}\right)+f\left(\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial h_{001}}\right)+c\left(\frac{\partial\sigma}{\partial g_{001}}+2\frac{\partial\sigma}{\partial c_{100}}\right)+\phi_{001}\frac{\partial\sigma}{\partial\phi_{101}}=0 \end{aligned}\right\} \quad . \quad (\mathrm{II}_3),$$

from the coefficients of $\xi_{101}$, $\eta_{101}$, $\zeta_{101}$ respectively; the three equations

$$\left.\begin{aligned} a\frac{\partial\sigma}{\partial h_{010}}+2h\frac{\partial\sigma}{\partial b_{010}}+g\frac{\partial\sigma}{\partial f_{010}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{020}}=0\\ h\frac{\partial\sigma}{\partial h_{010}}+2b\frac{\partial\sigma}{\partial b_{010}}+f\frac{\partial\sigma}{\partial f_{010}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{020}}=0\\ g\frac{\partial\sigma}{\partial h_{010}}+2f\frac{\partial\sigma}{\partial b_{010}}+c\frac{\partial\sigma}{\partial f_{010}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{020}}=0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{II}_4),$$

from the coefficients of $\xi_{020}$, $\eta_{020}$, $\zeta_{020}$ respectively; the three equations

$$\left.\begin{aligned} a\left(\frac{\partial\sigma}{\partial h_{001}}+\frac{\partial\sigma}{\partial g_{010}}\right)+h\left(2\frac{\partial\sigma}{\partial b_{001}}+\frac{\partial\sigma}{\partial f_{010}}\right)+g\left(\frac{\partial\sigma}{\partial f_{001}}+2\frac{\partial\sigma}{\partial c_{010}}\right)+\phi_{100}\frac{\partial\sigma}{\partial\phi_{011}}=0\\ h\left(\frac{\partial\sigma}{\partial h_{001}}+\frac{\partial\sigma}{\partial g_{010}}\right)+b\left(2\frac{\partial\sigma}{\partial b_{001}}+\frac{\partial\sigma}{\partial f_{010}}\right)+f\left(\frac{\partial\sigma}{\partial f_{001}}+2\frac{\partial\sigma}{\partial c_{010}}\right)+\phi_{010}\frac{\partial\sigma}{\partial\phi_{011}}=0\\ g\left(\frac{\partial\sigma}{\partial h_{001}}+\frac{\partial\sigma}{\partial g_{010}}\right)+f\left(2\frac{\partial\sigma}{\partial b_{001}}+\frac{\partial\sigma}{\partial f_{010}}\right)+c\left(\frac{\partial\sigma}{\partial f_{001}}+2\frac{\partial\sigma}{\partial c_{010}}\right)+\phi_{001}\frac{\partial\sigma}{\partial\phi_{011}}=0 \end{aligned}\right\} \; . \; (\mathrm{II}_5),$$

from the coefficients of $\xi_{011}$, $\eta_{011}$, $\zeta_{011}$ respectively; the three equations

$$\left.\begin{aligned} a\frac{\partial\sigma}{\partial g_{001}} + h\frac{\partial\sigma}{\partial f_{001}} + 2g\frac{\partial\sigma}{\partial c_{001}} + \phi_{100}\frac{\partial\sigma}{\partial\phi_{002}} = 0 \\ h\frac{\partial\sigma}{\partial g_{001}} + b\frac{\partial\sigma}{\partial f_{001}} + 2f\frac{\partial\sigma}{\partial c_{001}} + \phi_{010}\frac{\partial\sigma}{\partial\phi_{002}} = 0 \\ g\frac{\partial\sigma}{\partial g_{001}} + f\frac{\partial\sigma}{\partial f_{001}} + 2c\frac{\partial\sigma}{\partial c_{001}} + \phi_{001}\frac{\partial\sigma}{\partial\phi_{002}} = 0 \end{aligned}\right\} \quad \ldots\ldots \quad (\mathrm{II}_6),$$

from the coefficients of $\xi_{002}$, $\eta_{002}$, $\zeta_{002}$ respectively; the six equations

$$\begin{aligned} \Delta_1(\sigma) = {} & 2h\frac{\partial\sigma}{\partial a} + b\frac{\partial\sigma}{\partial h} + f\frac{\partial\sigma}{\partial g} + 2\phi_{110}\frac{\partial\sigma}{\partial\phi_{200}} + \phi_{020}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{011}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{010}\frac{\partial\sigma}{\partial\phi_{100}} \\ & + (a_{010} + 2h_{100})\frac{\partial\sigma}{\partial a_{100}} + 2h_{010}\frac{\partial\sigma}{\partial a_{010}} + 2h_{001}\frac{\partial\sigma}{\partial a_{001}} + b_{010}\frac{\partial\sigma}{\partial b_{100}} + c_{010}\frac{\partial\sigma}{\partial c_{100}} \\ & + f_{010}\frac{\partial\sigma}{\partial f_{100}} + (g_{010} + f_{100})\frac{\partial\sigma}{\partial g_{100}} + f_{010}\frac{\partial\sigma}{\partial g_{010}} + f_{001}\frac{\partial\sigma}{\partial g_{001}} \\ & \qquad + (h_{010} + b_{100})\frac{\partial\sigma}{\partial h_{100}} + b_{010}\frac{\partial\sigma}{\partial h_{010}} + b_{001}\frac{\partial\sigma}{\partial h_{001}} = 0, \end{aligned}$$

$$\begin{aligned} \Delta_2(\sigma) = {} & 2g\frac{\partial\sigma}{\partial a} + f\frac{\partial\sigma}{\partial h} + c\frac{\partial\sigma}{\partial g} + 2\phi_{101}\frac{\partial\sigma}{\partial\phi_{200}} + \phi_{011}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{002}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{001}\frac{\partial\sigma}{\partial\phi_{100}} \\ & + (a_{001} + 2g_{100})\frac{\partial\sigma}{\partial a_{100}} + 2g_{010}\frac{\partial\sigma}{\partial a_{010}} + 2g_{001}\frac{\partial\sigma}{\partial a_{001}} + b_{001}\frac{\partial\sigma}{\partial b_{100}} + c_{001}\frac{\partial\sigma}{\partial c_{100}} \\ & + f_{001}\frac{\partial\sigma}{\partial f_{100}} + (g_{001} + c_{100})\frac{\partial\sigma}{\partial g_{100}} + c_{010}\frac{\partial\sigma}{\partial g_{010}} + c_{001}\frac{\partial\sigma}{\partial g_{001}} \\ & \qquad + (h_{001} + f_{100})\frac{\partial\sigma}{\partial h_{100}} + f_{010}\frac{\partial\sigma}{\partial h_{010}} + f_{001}\frac{\partial\sigma}{\partial h_{001}} = 0, \end{aligned}$$

$$\begin{aligned} \Delta_3(\sigma) = {} & a\frac{\partial\sigma}{\partial h} + 2h\frac{\partial\sigma}{\partial b} + g\frac{\partial\sigma}{\partial f} + 2\phi_{110}\frac{\partial\sigma}{\partial\phi_{020}} + \phi_{200}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{101}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{100}\frac{\partial\sigma}{\partial\phi_{010}} \\ & + a_{100}\frac{\partial\sigma}{\partial a_{010}} + 2h_{100}\frac{\partial\sigma}{\partial b_{100}} + (b_{100} + 2h_{010})\frac{\partial\sigma}{\partial b_{010}} + 2h_{001}\frac{\partial\sigma}{\partial b_{001}} + c_{100}\frac{\partial\sigma}{\partial c_{010}} \\ & + g_{100}\frac{\partial\sigma}{\partial f_{100}} + (f_{100} + g_{010})\frac{\partial\sigma}{\partial f_{010}} + g_{001}\frac{\partial\sigma}{\partial f_{001}} + g_{100}\frac{\partial\sigma}{\partial g_{010}} \\ & \qquad + a_{100}\frac{\partial\sigma}{\partial h_{100}} + (a_{010} + h_{100})\frac{\partial\sigma}{\partial h_{010}} + a_{001}\frac{\partial\sigma}{\partial h_{001}} = 0, \end{aligned}$$

$$\begin{aligned} \Delta_4(\sigma) = {} & g\frac{\partial\sigma}{\partial h} + 2f\frac{\partial\sigma}{\partial b} + c\frac{\partial\sigma}{\partial f} + 2\phi_{011}\frac{\partial\sigma}{\partial\phi_{020}} + \phi_{101}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{002}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{001}\frac{\partial\sigma}{\partial\phi_{010}} \\ & + a_{001}\frac{\partial\sigma}{\partial a_{010}} + 2f_{100}\frac{\partial\sigma}{\partial b_{100}} + (b_{001} + 2f_{010})\frac{\partial\sigma}{\partial b_{010}} + 2f_{001}\frac{\partial\sigma}{\partial b_{001}} + c_{001}\frac{\partial\sigma}{\partial c_{010}} \\ & + c_{100}\frac{\partial\sigma}{\partial f_{100}} + (f_{001} + c_{010})\frac{\partial\sigma}{\partial f_{010}} + c_{001}\frac{\partial\sigma}{\partial f_{001}} + g_{001}\frac{\partial\sigma}{\partial g_{010}} \\ & \qquad + g_{100}\frac{\partial\sigma}{\partial h_{100}} + (g_{010} + h_{001})\frac{\partial\sigma}{\partial h_{010}} + g_{001}\frac{\partial\sigma}{\partial h_{001}} = 0, \end{aligned}$$



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$$\Delta_5(\sigma) = a\frac{\partial\sigma}{\partial g} + h\frac{\partial\sigma}{\partial f} + 2g\frac{\partial\sigma}{\partial c} + 2\phi_{101}\frac{\partial\sigma}{\partial\phi_{002}} + \phi_{200}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{110}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{100}\frac{\partial\sigma}{\partial\phi_{001}}$$
$$+ a_{100}\frac{\partial\sigma}{\partial a_{001}} + b_{100}\frac{\partial\sigma}{\partial b_{001}} + 2g_{100}\frac{\partial\sigma}{\partial c_{100}} + 2g_{010}\frac{\partial\sigma}{\partial c_{010}} + (c_{100} + 2g_{001})\frac{\partial\sigma}{\partial c_{100}}$$
$$+ h_{100}\frac{\partial\sigma}{\partial f_{100}} + h_{010}\frac{\partial\sigma}{\partial f_{010}} + (f_{100} + h_{001})\frac{\partial\sigma}{\partial f_{001}} + a_{100}\frac{\partial\sigma}{\partial g_{100}}$$
$$+ a_{010}\frac{\partial\sigma}{\partial g_{010}} + (a_{001} + g_{100})\frac{\partial\sigma}{\partial g_{001}} + h_{100}\frac{\partial\sigma}{\partial h_{001}} = 0,$$

$$\Delta_6(\sigma) = h\frac{\partial\sigma}{\partial g} + b\frac{\partial\sigma}{\partial f} + 2f\frac{\partial\sigma}{\partial c} + 2\phi_{011}\frac{\partial\sigma}{\partial\phi_{002}} + \phi_{110}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{020}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{010}\frac{\partial\sigma}{\partial\phi_{001}}$$
$$+ a_{010}\frac{\partial\sigma}{\partial a_{001}} + b_{010}\frac{\partial\sigma}{\partial b_{001}} + 2f_{100}\frac{\partial\sigma}{\partial c_{100}} + 2f_{010}\frac{\partial\sigma}{\partial c_{010}} + (c_{010} + 2f_{001})\frac{\partial\sigma}{\partial c_{100}}$$
$$+ b_{100}\frac{\partial\sigma}{\partial f_{100}} + b_{010}\frac{\partial\sigma}{\partial f_{010}} + (b_{001} + f_{010})\frac{\partial\sigma}{\partial f_{001}} + h_{100}\frac{\partial\sigma}{\partial g_{100}}$$
$$+ h_{010}\frac{\partial\sigma}{\partial g_{010}} + (h_{001} + g_{010})\frac{\partial\sigma}{\partial g_{001}} + h_{010}\frac{\partial\sigma}{\partial h_{001}} = 0,$$

from the coefficients of $\eta_{100}$, $\zeta_{100}$, $\xi_{010}$, $\zeta_{010}$, $\xi_{001}$, $\eta_{001}$ respectively; and the three equations

$$\Delta_7(\sigma) = 2a\frac{\partial\sigma}{\partial a} + h\frac{\partial\sigma}{\partial h} + g\frac{\partial\sigma}{\partial g} + 2\phi_{200}\frac{\partial\sigma}{\partial\phi_{200}} + \phi_{110}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{101}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}$$
$$+ 3a_{100}\frac{\partial\sigma}{\partial a_{100}} + 2a_{010}\frac{\partial\sigma}{\partial a_{010}} + 2a_{001}\frac{\partial\sigma}{\partial a_{001}} + b_{100}\frac{\partial\sigma}{\partial b_{100}} + c_{100}\frac{\partial\sigma}{\partial c_{100}}$$
$$+ f_{100}\frac{\partial\sigma}{\partial f_{100}} + 2g_{100}\frac{\partial\sigma}{\partial g_{100}} + g_{010}\frac{\partial\sigma}{\partial g_{010}} + g_{001}\frac{\partial\sigma}{\partial g_{001}} + 2h_{100}\frac{\partial\sigma}{\partial h_{100}}$$
$$+ h_{010}\frac{\partial\sigma}{\partial h_{010}} + h_{001}\frac{\partial\sigma}{\partial h_{001}} = \mu\sigma,$$

$$\Delta_8(\sigma) = h\frac{\partial\sigma}{\partial h} + 2b\frac{\partial\sigma}{\partial b} + f\frac{\partial\sigma}{\partial f} + 2\phi_{020}\frac{\partial\sigma}{\partial\phi_{020}} + \phi_{110}\frac{\partial\sigma}{\partial\phi_{110}} + \phi_{011}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{010}\frac{\partial\sigma}{\partial\phi_{010}}$$
$$+ a_{010}\frac{\partial\sigma}{\partial a_{010}} + 2b_{100}\frac{\partial\sigma}{\partial b_{100}} + 3b_{010}\frac{\partial\sigma}{\partial b_{010}} + 2b_{001}\frac{\partial\sigma}{\partial b_{001}} + c_{010}\frac{\partial\sigma}{\partial c_{010}}$$
$$+ f_{100}\frac{\partial\sigma}{\partial f_{100}} + 2f_{010}\frac{\partial\sigma}{\partial f_{010}} + f_{001}\frac{\partial\sigma}{\partial f_{001}} + g_{010}\frac{\partial\sigma}{\partial g_{010}} + h_{100}\frac{\partial\sigma}{\partial h_{100}}$$
$$+ 2h_{010}\frac{\partial\sigma}{\partial h_{010}} + h_{001}\frac{\partial\sigma}{\partial h_{001}} = \mu\sigma,$$

$$\Delta_9(\sigma) = g\frac{\partial\sigma}{\partial g} + f\frac{\partial\sigma}{\partial f} + 2c\frac{\partial\sigma}{\partial c} + 2\phi_{002}\frac{\partial\sigma}{\partial\phi_{002}} + \phi_{101}\frac{\partial\sigma}{\partial\phi_{101}} + \phi_{011}\frac{\partial\sigma}{\partial\phi_{011}} + \phi_{001}\frac{\partial\sigma}{\partial\phi_{001}}$$
$$+ a_{001}\frac{\partial\sigma}{\partial a_{001}} + b_{001}\frac{\partial\sigma}{\partial b_{001}} + 2c_{100}\frac{\partial\sigma}{\partial c_{100}} + 2c_{010}\frac{\partial\sigma}{\partial c_{010}} + 3c_{001}\frac{\partial\sigma}{\partial c_{001}}$$
$$+ f_{100}\frac{\partial\sigma}{\partial f_{100}} + f_{010}\frac{\partial\sigma}{\partial f_{010}} + 2f_{001}\frac{\partial\sigma}{\partial f_{001}} + g_{100}\frac{\partial\sigma}{\partial g_{100}} + g_{010}\frac{\partial\sigma}{\partial g_{010}}$$
$$+ 2g_{001}\frac{\partial\sigma}{\partial g_{001}} + h_{001}\frac{\partial\sigma}{\partial h_{001}} = \mu\sigma,$$

from the coefficients of $\xi_{100}$, $\eta_{010}$, $\zeta_{001}$ respectively.

### *The simplest of the Invariants.*

8. The only differential invariant, which involves $a$, $b$, $c$, $f$, $g$, $h$, without any of their derivatives and without any of the derivatives of $\phi$, is $L^2$.

The equations $(II_1), \ldots, (II_6)$, as well as all equations arising from the higher derivatives of $\xi$, $\eta$, $\zeta$ are evanescent when no derivatives of $a$, $b$, $c$, $f$, $g$, $h$, $\phi$ occur. The six equations $\Delta_1(\sigma), \ldots, \Delta_6(\sigma)$ are satisfied uniquely by

$$\frac{2}{A}\frac{\partial\sigma}{\partial a} = \frac{2}{B}\frac{\partial\sigma}{\partial b} = \frac{2}{C}\frac{\partial\sigma}{\partial c} = \frac{1}{F}\frac{\partial\sigma}{\partial f} = \frac{1}{G}\frac{\partial\sigma}{\partial g} = \frac{1}{H}\frac{\partial\sigma}{\partial h}$$
$$= \rho,$$

say; and the three equations $\Delta_7(\sigma)$, $\Delta_8(\sigma)$, $\Delta_9(\sigma)$ are then satisfied uniquely by

$$\rho = \mu\frac{\sigma}{L^2}.$$

By effecting quadratures in the relation

$$d\sigma = \frac{\partial\sigma}{\partial a}da + \ldots \qquad + \frac{\partial\sigma}{\partial h}dh,$$

we find that $\sigma$ is a constant multiple of $L^\mu$. The lowest power of L, which is rational in $a$, $b$, $c$, $f$, $g$, $h$, is $L^2$; we therefore take $L^2$ as the one differential invariant of the specified character.

9. There is no proper differential invariant of the first order in the quantities $a$, $b$, $c$, $f$, $g$, $h$ alone, that is, there is no invariant (other than $L^2$) involving these quantities and their first derivatives but no other variable magnitudes.

Let any such invariant, if it exists, be denoted by $\sigma$. The equations which arise through derivatives of $\xi$, $\eta$, $\zeta$ of order higher than the second are evanescent. From the equations $(II_1)$, $(II_4)$, $II_6)$, we have

$$\frac{\partial\sigma}{\partial a_{100}} = \frac{\partial\sigma}{\partial h_{100}} = \frac{\partial\sigma}{\partial g_{100}} = 0,$$
$$\frac{\partial\sigma}{\partial h_{010}} = \frac{\partial\sigma}{\partial b_{010}} = \frac{\partial\sigma}{\partial f_{010}} = 0,$$
$$\frac{\partial\sigma}{\partial g_{001}} = \frac{\partial\sigma}{\partial f_{001}} = \frac{\partial\sigma}{\partial c_{001}} = 0;$$

and from the equations ($II_2$), ($II_3$), ($II_5$), we have

$$2\frac{\partial\sigma}{\partial a_{010}}+\frac{\partial\sigma}{\partial h_{100}}=0=2\frac{\partial\sigma}{\partial b_{100}}+\frac{\partial\sigma}{\partial h_{010}}=\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial g_{010}},$$

$$2\frac{\partial\sigma}{\partial a_{001}}+\frac{\partial\sigma}{\partial g_{100}}=0=2\frac{\partial\sigma}{\partial c_{100}}+\frac{\partial\sigma}{\partial g_{001}}=\frac{\partial\sigma}{\partial f_{100}}+\frac{\partial\sigma}{\partial h_{001}},$$

$$2\frac{\partial\sigma}{\partial b_{001}}+\frac{\partial\sigma}{\partial f_{010}}=0=2\frac{\partial\sigma}{\partial c_{010}}+\frac{\partial\sigma}{\partial f_{001}}=\frac{\partial\sigma}{\partial g_{010}}+\frac{\partial\sigma}{\partial h_{001}}.$$

These equations show that no one of the first derivatives of $a$, $b$, $c$, $f$, $g$, $h$ occur in the hypothetical invariant; that is, there is no proper differential invariant of the first order in the quantities $a$, $b$, $c$, $f$, $g$, $h$ alone.

10. Next, are there any differential invariants, which involve $a$, $b$, $c$, $f$, $g$, $h$ but no derivatives of these quantities and which also involve derivatives of a single function $\phi$ of the first order but none of higher orders?

The nine characteristic equations are

$$2h\frac{\partial\sigma}{\partial a}+b\frac{\partial\sigma}{\partial h}+f\frac{\partial\sigma}{\partial g}+\phi_{010}\frac{\partial}{\partial\phi_{100}}=0,$$

$$2g\frac{\partial\sigma}{\partial a}+f\frac{\partial\sigma}{\partial h}+c\frac{\partial\sigma}{\partial g}+\phi_{001}\frac{\partial}{\partial\phi_{100}}=0,$$

$$a\frac{\partial\sigma}{\partial h}+2h\frac{\partial\sigma}{\partial b}+g\frac{\partial\sigma}{\partial f}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$g\frac{\partial\sigma}{\partial h}+2f\frac{\partial\sigma}{\partial b}+c\frac{\partial\sigma}{\partial f}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$a\frac{\partial\sigma}{\partial g}+h\frac{\partial\sigma}{\partial f}+2g\frac{\partial\sigma}{\partial c}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{001}}=0,$$

$$h\frac{\partial\sigma}{\partial g}+b\frac{\partial\sigma}{\partial f}+2f\frac{\partial\sigma}{\partial c}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{001}}=0,$$

$$2a\frac{\partial\sigma}{\partial a}+g\frac{\partial\sigma}{\partial g}-f\frac{\partial\sigma}{\partial f}-2b\frac{\partial\sigma}{\partial b}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}-\phi_{010}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$2a\frac{\partial\sigma}{\partial a}+h\frac{\partial\sigma}{\partial h}-f\frac{\partial\sigma}{\partial f}-2c\frac{\partial\sigma}{\partial c}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}-\phi_{001}\frac{\partial\sigma}{\partial\phi_{001}}=0,$$

$$2\left(a\frac{\partial\sigma}{\partial a}+b\frac{\partial\sigma}{\partial c}+c\frac{\partial\sigma}{\partial c}+f\frac{\partial\sigma}{\partial f}+g\frac{\partial\sigma}{\partial g}+h\frac{\partial\sigma}{\partial h}\right)$$
$$+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{010}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{001}}=3\mu\sigma.$$

The last three equations replace $\Delta_7(\sigma) = \mu\sigma$, $\Delta_8(\sigma) = \mu\sigma$, $\Delta_9(\sigma) = \mu\sigma$, being $\Delta_7(\sigma) - \Delta_8(\sigma) = 0$, $\Delta_7(\sigma) - \Delta_9(\sigma) = 0$, $\Delta_7(\sigma) + \Delta_8(\sigma) + \Delta_9(\sigma) = 3\mu\sigma$.

Inspection of the first eight equations shows that they are the differential equations of the invariants and the contravariants of the ternary quadratic

$$(a, b, c, f, g, h \between X, Y, Z)^2,$$

the contragradient variables being $\phi_{100}$, $\phi_{010}$, $\phi_{001}$. Let $t$ be such an invariant or the leading coefficient of such a contravariant, that is, the coefficient of the highest power of $\phi_{100}$; in the latter case, the contravariant is uniquely determinate when $t$ is known. Now $t$ satisfies the five equations

$$2h\frac{\partial t}{\partial a} + b\frac{\partial t}{\partial h} + f\frac{\partial t}{\partial g} = 0,$$

$$2g\frac{\partial t}{\partial a} + f\frac{\partial t}{\partial h} + c\frac{\partial t}{\partial g} = 0,$$

$$g\frac{\partial t}{\partial h} + 2f\frac{\partial t}{\partial b} + c\frac{\partial t}{\partial f} = 0,$$

$$h\frac{\partial t}{\partial g} + b\frac{\partial t}{\partial f} + 2f\frac{\partial t}{\partial c} = 0,$$

$$2b\frac{\partial t}{\partial b} - 2c\frac{\partial t}{\partial c} + h\frac{\partial t}{\partial h} - g\frac{\partial t}{\partial g} = 0.$$

This is a complete Jacobian system. It apparently contains five members: but the fifth equation is a linear combination of the first four, and therefore the system really consists of four equations. It involves six variables, viz., $a, b, c, f, g, h$; and therefore it possesses two independent solutions. One of these is $L^2$; the other is easily seen to be $bc - f^2$, $=$ A.

Returning now to the system of nine equations, and bearing in mind the fact that the first eight possess two solutions, one being $L^2$ and the other being a contravariant of the ternary quadratic which has A for its leading coefficient, we have the contravariant in the well-known form

$$\Theta = (A, B, C, F, G, H \between \phi_{100}, \phi_{010}, \phi_{001})^2.$$

The ninth equation is satisfied for any solution which is homogeneous in $a, b, c, f, g, h$ of degree $m$ and is also homogeneous in $\phi_{100}$, $\phi_{010}$, $\phi_{001}$ of degree $n$, provided

$$2m + n = 3\mu.$$

Hence the value of $\mu$ for $\Theta$ is 2.

There is thus a single absolute differential invariant of the type specified; its value is

$$\Theta L^{-2}.$$

11. Proceeding as in § 9, we can similarly prove that there is no proper differential invariant of the first order in the quantities $a$, $b$, $c$, $f$, $g$, $h$, which involves the first derivatives of $\phi$, that is, that $L^2$ and $\Theta$ are the only invariants which involve the first derivatives of $\phi$, the quantities $a$, $b$, $c$, $f$, $g$, $h$, and their derivatives of the first order.

### *Invariants of the Second Order.*

12. We now proceed to consider the aggregate of algebraically independent differential invariants, which involve $a$, $b$, $c$, $f$, $g$, $h$ and their first derivatives, and which also involve derivatives of $\phi$ up to the second order inclusive. It has been proved that when no derivative of $\phi$ occurs, there is only a single such invariant, viz., $L^2$; and that there is one such invariant involving the first derivatives of $\phi$, viz., $\Theta$.

The equations characteristic to the invariants are the set given in § 7; their number is $18 + 6 + 3 = 27$. Taking the last three in the equivalent form

$$\Delta_7(\sigma) - \Delta_8(\sigma) = 0,$$
$$\Delta_7(\sigma) - \Delta_9(\sigma) = 0,$$
$$\Delta_7(\sigma) + \Delta_8(\sigma) + \Delta_9(\sigma) = 3\mu\sigma,$$

and associating the first two of these with the earlier $18 + 6$, we have 26 equations in all. They are linearly independent of one another; and they form a complete Jacobian system. The number of variables that occur is

6, for the quantities $a$, $b$, $c$, $f$, $g$, $h$,
+ 18, for their first derivatives,
+ 3, for the first derivatives of $\phi$,
+ 6, . . second . . . $\phi$,

being 33 in all. There are therefore 7 solutions common to the 26 equations; two of them are already known, being $L^2$ and $\Theta$; and therefore other five are required. It is manifest, from the manner in which these five have been selected, that each of them must involve second derivatives of $\phi$.

The remaining equation will be found to be satisfied for each of the five solutions by the appropriate determination of the index $\mu$ in each case: the actual determination is made in a simple manner, owing to properties of homogeneity possessed by the solution.

13. The mode of manipulating the equations, so as to obtain an algebraically complete aggregate of integrals, is similar to the mode in the corresponding investigation of the differential invariants of surfaces.

We begin by obtaining the proper number of independent integrals which belong to the set of eighteen equations $(II_1), \ldots, (II_6)$. This set is, in itself, a complete

Jacobian set; as it involves 33 variable magnitudes, it possesses 15 independent integrals. Of these 15, nine are easily seen to be

$$a,\ b,\ c,\ f,\ g,\ h,$$
$$\phi_{100},\ \phi_{010},\ \phi_{001}.$$

Other six are found to be

$$\left.\begin{aligned}
\mathfrak{a} &= 2L^2\phi_{200} - Pa_{100} - Q(2h_{100} - a_{010}) - R(2g_{100} - a_{001}) \\
\mathfrak{b} &= 2L^2\phi_{020} - P(2h_{010} - b_{100}) - Qb_{010} - R(2f_{010} - b_{001}) \\
\mathfrak{c} &= 2L^2\phi_{002} - P(2g_{001} - c_{100}) - Q(2f_{001} - c_{010}) - Rc_{001} \\
\mathfrak{f} &= 2L^2\phi_{011} - P(-f_{100} + g_{010} + h_{001}) - Qb_{001} - Rc_{010} \\
\mathfrak{g} &= 2L^2\phi_{101} - Pa_{001} - Q(f_{100} - g_{010} + h_{001}) - Rc_{100} \\
\mathfrak{h} &= 2L^2\phi_{110} - Pa_{010} - Qb_{100} - R(f_{100} + g_{010} - h_{001})
\end{aligned}\right\},$$

where

$$P = \begin{vmatrix} \phi_{100}, & h, & g \\ \phi_{010}, & b, & f \\ \phi_{001}, & f, & c \end{vmatrix} = A\phi_{100} + H\phi_{010} + G\phi_{001},$$

$$Q = \begin{vmatrix} \phi_{100}, & g, & a \\ \phi_{010}, & f, & h \\ \phi_{001}, & c, & g \end{vmatrix} = H\phi_{100} + B\phi_{010} + F\phi_{001},$$

$$R = \begin{vmatrix} \phi_{100}, & a, & h \\ \phi_{010}, & h, & b \\ \phi_{001}, & g, & f \end{vmatrix} = G\phi_{100} + F\phi_{010} + C\phi_{001}.$$

14. A way of constructing these six solutions will be sufficiently illustrated by giving an outline of the process for any one of them, say the first. The form of the differential equations suggests that they possess solutions, which are linear in the first derivatives of $a$, $b$, $c$, $f$, $g$, $h$ and in the second derivatives of $\phi$; we therefore postulate a solution in the form

$$\begin{aligned}
2L^2\phi_{200} &- \alpha a_{100} - \beta h_{100} - \gamma g_{100} - \kappa b_{100} - \lambda f_{100} - \mu c_{100} \\
&- \delta a_{010} - \zeta h_{010} - \theta g_{010} - \nu b_{010} - \rho f_{010} - \tau c_{010} \\
&- \epsilon a_{001} - \eta h_{001} - \iota g_{001} - \pi b_{001} - \chi f_{001} - \upsilon c_{001},
\end{aligned}$$

where the coefficients $\alpha, \ldots, \upsilon$ do not involve first derivatives of $a$, $b$, $c$, $f$, $g$, $h$ or second derivatives of $\phi$. In order that this expression may satisfy the three equations ($II_1$), we must have

$$2a\alpha + h\beta + g\gamma = 2L^2\phi_{100},$$
$$2h\alpha + b\beta + f\gamma = 2L^2\phi_{010},$$
$$2g\alpha + f\beta + c\gamma = 2L^2\phi_{001},$$

and therefore

$$\alpha = \mathrm{P}, \quad \beta = 2\mathrm{Q}, \quad \gamma = 2\mathrm{R}.$$

In order that the three equations ($\mathrm{II}_2$) may be satisfied, we must have

$$a(2\delta + \beta) + h(2\kappa + \zeta) + g(\lambda + \theta) = 0,$$
$$h(2\delta + \beta) + b(2\kappa + \zeta) + f(\lambda + \theta) = 0,$$
$$g(2\delta + \beta) + f(2\kappa + \zeta) + c(\lambda + \theta) = 0,$$

and therefore

$$2\delta + \beta = 0, \quad 2\kappa + \zeta = 0, \quad \lambda + \theta = 0.$$

The three equations ($\mathrm{II}_3$) similarly give

$$2\epsilon + \gamma = 0, \quad 2\mu + \iota = 0, \quad \lambda + \eta = 0\,;$$

the three equations ($\mathrm{II}_4$) give

$$\nu = 0, \quad \rho = 0, \quad \zeta = 0\,;$$

the three equations ($\mathrm{II}_5$) give

$$2\pi + \rho = 0, \quad 2\tau + \chi = 0, \quad \theta + \eta = 0\,;$$

and the three equations ($\mathrm{II}_6$) give

$$\upsilon = 0, \quad \chi = 0, \quad \iota = 0.$$

When these equations are solved, it appears that the only coefficients (other than $\alpha$, $\beta$, $\gamma$) which do not vanish are $\delta$ and $\epsilon$; their values are

$$\delta = -\mathrm{Q}, \quad \epsilon = -\mathrm{R}.$$

Inserting these values, we obtain the quantity which has been denoted by **a**. The other five solutions can be constructed in the same way.

15. It may be remarked that there exists a certain symmetry in the quantities already obtained (and in corresponding quantities of higher orders) which may be used for comparative verification of results obtained or for avoidance of long stretches of algebra by deducing new results from results obtained. The symmetry arises through the effects caused by interchanging $u$ and $v$, $v$ and $w$, $u$ and $w$, as follows:—

| | $a$ | $b$ | $c$ | $f$ | $g$ | $h$ | **a** | **b** | **c** | **f** | **g** | **h** | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $u$ and $v$ | $b$ | $a$ | $c$ | $g$ | $f$ | $h$ | **b** | **a** | **c** | **g** | **f** | **h** | Q | P | R |
| $v$ and $w$ | $a$ | $c$ | $b$ | $f$ | $h$ | $g$ | **a** | **c** | **b** | **f** | **h** | **g** | P | R | Q |
| $u$ and $w$ | $c$ | $b$ | $a$ | $h$ | $g$ | $f$ | **c** | **b** | **a** | **h** | **g** | **f** | R | Q | P |

Thus R is unaltered when $u$ and $v$ are interchanged; it is changed to Q, when $v$ and $w$ are interchanged, and to P, when $u$ and $w$ are interchanged. The interchanges of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ among one another, and likewise those of $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ among one another, caused by interchanges of the variables $u$, $v$, $w$, are indicated in the table; they could be used to deduce $\mathfrak{b}$ and $\mathfrak{c}$ when $\mathfrak{a}$ is known, and to deduce $\mathfrak{g}$ and $\mathfrak{h}$ when $\mathfrak{f}$ is known.

16. We have now to obtain five functional combinations (other than $L^2$ and $\Theta$) of the fifteen quantities $a$, $b$, $c$, $f$, $g$, $h$, $\phi_{100}$, $\phi_{010}$, $\phi_{001}$, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$, which satisfy the nine equations

$$\Delta_1(\sigma) = 0, \ldots, \quad \Delta_6(\sigma) = 0,$$

$$\Delta_7(\sigma) - \Delta_8(\sigma) = 0, \quad \Delta_7(\sigma) - \Delta_9(\sigma) = 0, \quad \Delta_7(\sigma) + \Delta_8(\sigma) + \Delta_9(\sigma) = 3\mu\sigma;$$

and the functional combinations which are required must contain some of the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$.

It is easy to verify the results in the following table:—

| | $\mathfrak{a}$ | $\mathfrak{b}$ | $\mathfrak{c}$ | $\mathfrak{f}$ | $\mathfrak{g}$ | $\mathfrak{h}$ |
|---|---|---|---|---|---|---|
| $\Delta_1$ | $2\mathfrak{h}$ | 0 | 0 | 0 | $\mathfrak{f}$ | $\mathfrak{b}$ |
| $\Delta_2$ | $2\mathfrak{g}$ | 0 | 0 | 0 | $\mathfrak{c}$ | $\mathfrak{f}$ |
| $\Delta_3$ | 0 | $2\mathfrak{h}$ | 0 | $\mathfrak{g}$ | 0 | $\mathfrak{a}$ |
| $\Delta_4$ | 0 | $2\mathfrak{f}$ | 0 | $\mathfrak{c}$ | 0 | $\mathfrak{g}$ |
| $\Delta_5$ | 0 | 0 | $2\mathfrak{g}$ | $\mathfrak{h}$ | $\mathfrak{a}$ | 0 |
| $\Delta_6$ | 0 | 0 | $2\mathfrak{f}$ | $\mathfrak{b}$ | | 0 |
| $\Delta_7$ | $4\mathfrak{a}$ | $2\mathfrak{b}$ | $2\mathfrak{c}$ | $2\mathfrak{f}$ | $3\mathfrak{g}$ | $3\mathfrak{h}$ |
| $\Delta_8$ | $2\mathfrak{a}$ | $4\mathfrak{b}$ | $2\mathfrak{c}$ | $3\mathfrak{f}$ | $2\mathfrak{g}$ | $3\mathfrak{h}$ |
| $\Delta_9$ | $2\mathfrak{a}$ | $2\mathfrak{b}$ | $4\mathfrak{c}$ | $3\mathfrak{f}$ | $3\mathfrak{g}$ | $2\mathfrak{h}$ |

which should be read $\Delta_1(\mathfrak{a}) = 2\mathfrak{h}$, $\Delta_1(\mathfrak{b}) = 0$, and so on. Hence now denoting by $\sigma$ any one of the five functional combinations of the fifteen arguments, the first eight of the equations take the form

$$2h\frac{\partial\sigma}{\partial a}+b\frac{\partial\sigma}{\partial h}+f\frac{\partial\sigma}{\partial g}+2\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{a}}+\mathfrak{b}\frac{\partial\sigma}{\partial\mathfrak{h}}+\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{g}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{100}}=0,$$

$$2g\frac{\partial\sigma}{\partial a}+f\frac{\partial\sigma}{\partial h}+c\frac{\partial\sigma}{\partial g}+2\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{a}}+\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{h}}+\mathfrak{c}\frac{\partial\sigma}{\partial\mathfrak{g}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{100}}=0,$$

$$a\frac{\partial\sigma}{\partial h}+2h\frac{\partial\sigma}{\partial b}+g\frac{\partial\sigma}{\partial f}+\mathfrak{a}\frac{\partial\sigma}{\partial\mathfrak{h}}+2\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{b}}+\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{f}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$g\frac{\partial\sigma}{\partial h}+2f\frac{\partial\sigma}{\partial b}+c\frac{\partial\sigma}{\partial f}+\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{h}}+2\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{b}}+\mathfrak{c}\frac{\partial\sigma}{\partial\mathfrak{f}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$a\frac{\partial\sigma}{\partial g}+h\frac{\partial\sigma}{\partial f}+2g\frac{\partial\sigma}{\partial c}+\mathfrak{a}\frac{\partial\sigma}{\partial\mathfrak{g}}+\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{f}}+2\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{c}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{001}}=0,$$

$$h\frac{\partial\sigma}{\partial g}+b\frac{\partial\sigma}{\partial f}+2f\frac{\partial\sigma}{\partial c}+\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{g}}+\mathfrak{b}\frac{\partial\sigma}{\partial\mathfrak{f}}+2\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{c}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{001}}=0,$$

$$2\left(a\frac{\partial\sigma}{\partial a}-b\frac{\partial\sigma}{\partial b}\right)+g\frac{\partial\sigma}{\partial g}-f\frac{\partial\sigma}{\partial f}+2\left(\mathfrak{a}\frac{\partial\sigma}{\partial\mathfrak{a}}-\mathfrak{b}\frac{\partial\sigma}{\partial\mathfrak{b}}\right)+\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{g}}-\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{f}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}-\phi_{010}\frac{\partial\sigma}{\partial\phi_{010}}=0,$$

$$2\left(a\frac{\partial\sigma}{\partial a}-c\frac{\partial\sigma}{\partial c}\right)+h\frac{\partial\sigma}{\partial h}-f\frac{\partial\sigma}{\partial f}+2\left(\mathfrak{a}\frac{\partial\sigma}{\partial\mathfrak{a}}-\mathfrak{c}\frac{\partial\sigma}{\partial\mathfrak{c}}\right)+\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{h}}-\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{f}}+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}-\phi_{001}\frac{\partial\sigma}{\partial\phi_{001}}=0.$$

The ninth equation takes the form

$$\begin{aligned}2\left(a\frac{\partial\sigma}{\partial a}+b\frac{\partial\sigma}{\partial b}+c\frac{\partial\sigma}{\partial c}+f\frac{\partial\sigma}{\partial f}+g\frac{\partial\sigma}{\partial g}+h\frac{\partial\sigma}{\partial h}\right)\\+8\left(\mathfrak{a}\frac{\partial\sigma}{\partial\mathfrak{a}}+\mathfrak{b}\frac{\partial\sigma}{\partial\mathfrak{b}}+\mathfrak{c}\frac{\partial\sigma}{\partial\mathfrak{c}}+\mathfrak{f}\frac{\partial\sigma}{\partial\mathfrak{f}}+\mathfrak{g}\frac{\partial\sigma}{\partial\mathfrak{g}}+\mathfrak{h}\frac{\partial\sigma}{\partial\mathfrak{h}}\right)\\+\phi_{100}\frac{\partial\sigma}{\partial\phi_{100}}+\phi_{010}\frac{\partial\sigma}{\partial\phi_{010}}+\phi_{001}\frac{\partial\sigma}{\partial\phi_{001}}=3\mu\sigma.\end{aligned}$$

Inspection of the first eight equations in this form, coupled with the limitation that some at least of the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ occur, shews that they are the differential equations of those invariants and contravariants of the simultaneous ternary quadratics

$$(a, b, c, f, g, h\between X, Y, Z)^2,$$

$$(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h}\between X, Y, Z)^2,$$

which involve the coefficients of the second of these quadratics, and that the contragradient variables (when they occur) are $\phi_{100}$, $\phi_{010}$, $\phi_{001}$. The ninth equation is satisfied when the index $\mu$ is appropriately determined as follows: let $\sigma$ be homogeneous in $\phi_{100}$, $\phi_{010}$, $\phi_{001}$ of degree $n$, in $a, b, c, f, g, h$ of degree $m$, in $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ of degree $l$; then

$$8l+2m+n=3\mu.$$

Now the complete system (being the asyzygetically complete system and not merely that which is algebraically complete) of two ternary quadratics is known;* it thus enables us to select the five concomitants required which, it is to be remembered, are to involve the quantities $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$. Let

$$\mathbf{A} = \mathbf{bc} - \mathbf{f}^2, \qquad \mathbf{B} = \mathbf{ca} - \mathbf{g}^2, \qquad \mathbf{C} = \mathbf{ab} - \mathbf{h}^2,$$
$$\mathbf{F} = \mathbf{gh} - \mathbf{af}, \qquad \mathbf{G} = \mathbf{hf} - \mathbf{bg}, \qquad \mathbf{H} = \mathbf{fg} - \mathbf{ch},$$

$$\mathfrak{A} = b\mathbf{c} + \mathbf{b}c - 2f\mathbf{f}, \qquad \mathfrak{F} = g\mathbf{h} + \mathbf{g}h - a\mathbf{f} - \mathbf{a}f,$$
$$\mathfrak{B} = c\mathbf{a} + \mathbf{c}a - 2g\mathbf{g}, \qquad \mathfrak{G} = h\mathbf{f} + \mathbf{h}f - b\mathbf{g} - \mathbf{b}g,$$
$$\mathfrak{C} = a\mathbf{b} + \mathbf{a}b - 2h\mathbf{h}, \qquad \mathfrak{H} = f\mathbf{g} + \mathbf{f}g - c\mathbf{h} - \mathbf{c}h;$$

and, in order to exhibit the relation of the concomitants to the two quadractic forms, let

$$L^2 = \Delta_1, \qquad \Theta = \Theta_1.$$

Then the five quantities required are

$$\Theta_{12} = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \phi_{100}, \phi_{010}, \phi_{001})^2,$$

$$\Theta_2 = (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{G}, \mathbf{H} \between \phi_{100}, \phi_{010}, \phi_{001})^2,$$

$$\Delta_{12} = A\mathbf{a} + B\mathbf{b} + C\mathbf{c} + 2F\mathbf{f} + 2G\mathbf{g} + 2H\mathbf{h},$$

$$\Delta_{21} = \mathbf{A}a + \mathbf{B}b + \mathbf{C}c + 2\mathbf{F}f + 2\mathbf{G}g + 2\mathbf{H}h,$$

$$\Delta_2 = \begin{vmatrix} \mathbf{a} & \mathbf{h} & \mathbf{g} \\ \mathbf{h} & \mathbf{b} & \mathbf{f} \\ \mathbf{g} & \mathbf{f} & \mathbf{c} \end{vmatrix}.$$

The respective values of $\mu$, as determined from the relation $8l + 2m + n = 3\mu$, are easily found: we have

$$\begin{array}{lll} \text{Index } 4, & \Theta_{12}, & \Delta_{12}; \\ \quad\;\; 6, & \Theta_2, & \Delta_{21}; \\ \quad\;\; 8, & \Delta_2. & \end{array}$$

We have already seen that the index of $\Theta_1 \,(= \Theta)$ is 2 and the index of $\Delta_1 \,(= L^2)$ is 2.

Hence *an aggregate of algebraically independent absolute invariants, which involve a, b, c, f, g, h and their first derivatives and which also involve derivatives of a single function $\phi$ up to the second order inclusive, is composed of*

$$\frac{\Theta_1}{\Delta_1}, \quad \frac{\Theta_{12}}{\Delta_1^{\,2}}, \quad \frac{\Delta_{12}}{\Delta_1^{\,2}}, \quad \frac{\Theta_2}{\Delta_1^{\,3}}, \quad \frac{\Delta_{21}}{\Delta_1^{\,3}}, \quad \frac{\Delta_2}{\Delta_1^{\,4}};$$

* CLEBSCH, 'Vorlesungen über Geometrie,' vol. 1, p. 290: the construction of the system is due to GORDAN.

*the last five involve derivatives of $\phi$ of the second order as well as derivatives of the first order, and the first involves derivatives of $\phi$ of the first order alone.*

Every other differential invariant, involving derivatives of $\phi$ of order not higher than the second, together with the quantities $a, b, c, f, g, h$ and their derivatives of the first order, is expressible in terms of the members of this aggregate. Such a differential invariant is provided by the discriminant of the quantity $\Theta_{12}$, and its expression is

$$\nabla = \begin{vmatrix} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C} \end{vmatrix}.$$

It is not difficult to prove that

$$\nabla = \Delta_{12}\Delta_{21} - \Delta_1\Delta_2.$$

17. It is also convenient, in view of the expressions for the differential invariants of the third order about to be considered, to give the umbral symbolical expressions for these invariants just considered. We write

$$\phi_{100} = u_1, \quad \phi_{010} = u_2, \quad \phi_{001} = u_3,$$
$$\mathrm{X} = x_1, \quad \mathrm{Y} = x_2, \quad \mathrm{Z} = x_3.$$

In connection with the first ternary quadratic, we introduce sets of umbral symbols $a_1, a_2, a_3$; $b_1, b_2, b_3$; and so on. In connection with the second ternary quadratic, we introduce sets of umbral symbols $a'_1, a'_2, a'_3$; $b'_1, b'_2, b'_3$; and so on. Then we write

$$(a, b, c, f, g, h \between \mathrm{X}, \mathrm{Y}, \mathrm{Z})^2 = a_x{}^2 = b_x{}^2 = \ldots,$$
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h} \between \mathrm{X}, \mathrm{Y}, \mathrm{Z})^2 = a'_x{}^2 = b'_x{}^2 = \ldots;$$

and the seven relative differential invariants are

$$\Delta_1 = \tfrac{1}{6}(abc)^2,$$
$$\Theta_1 = \tfrac{1}{2}(abu)^2,$$
$$\Theta_{12} = (aa'u)^2,$$
$$\Theta_2 = \tfrac{1}{2}(a'b'u)^2,$$
$$\Delta_{12} = \tfrac{1}{2}(a'ab)^2,$$
$$\Delta_{21} = \tfrac{1}{2}(aa'b)^2,$$
$$\Delta_2 = \tfrac{1}{6}(a'b'c')^2.$$

It is known* that another pure contravariant is possessed by two ternary quadratics; its symbolical expression is

$$(aa'u)(ab'c')(a'bc)(bcu)(b'c'u).$$

* CLEBSCH, 'Vorlesungen über Geometrie,' p. 291.

Denoting this by D, we ought to be able to express D algebraically in terms of the seven preceding forms; as a matter of fact, $D^2$ is a polynomial combination of the forms.

18. The preceding determination of the differential invariants of the specified order has been based upon a knowledge of the complete system of concomitants of two ternary quadratics. When we pass to higher orders, the last stage in the determination of the differential invariants could be completed without further calculation, if we knew the complete system of concomitants of certain simultaneous ternary quantics some of which are of order higher than two. But, in general, such knowledge is not at present possessed; in its absence, some other method of attaining the end is necessary. Such a method can be devised in connection with the differential equations; as applied to the two quadratics, it is as follows.

As has been pointed out, the equations determine the invariants and the contravariants of two simultaneous quadratics, the contragradient variables being $\phi_{100}$, $\phi_{010}$, $\phi_{001}$. Let such an one be

$$\sigma = t\phi_{100}{}^{n} + nt_1\phi_{100}{}^{n-1}\phi_{010} + nt_2\phi_{100}{}^{n-1}\phi_{001} + \ldots,$$

where $t$, $t_1$, $t_2$, ... are independent of $\phi_{100}$, $\phi_{010}$, $\phi_{001}$; and $n$ is a whole number, which in the case of an invariant is zero, and which, when $t$ is known, can always be determined by inspection. Then when this value of $\sigma$ is substituted in the ordinary way in the first six equations, it appears that $t$ satisfies the four equations

$$e_1(t) = 2h\frac{\partial t}{\partial a} + b\frac{\partial t}{\partial h} + f\frac{\partial t}{\partial g} + 2\mathbf{h}\frac{\partial t}{\partial \mathbf{a}} + \mathbf{b}\frac{\partial t}{\partial \mathbf{h}} + \mathbf{f}\frac{\partial t}{\partial \mathbf{g}} = 0,$$

$$e_2(t) = 2g\frac{\partial t}{\partial a} + f\frac{\partial t}{\partial h} + c\frac{\partial t}{\partial g} + 2\mathbf{g}\frac{\partial t}{\partial \mathbf{a}} + \mathbf{f}\frac{\partial t}{\partial \mathbf{h}} + \mathbf{c}\frac{\partial t}{\partial \mathbf{g}} = 0,$$

$$e_3(t) = g\frac{\partial t}{\partial h} + 2f\frac{\partial t}{\partial b} + c\frac{\partial t}{\partial f} + \mathbf{g}\frac{\partial t}{\partial \mathbf{h}} + 2\mathbf{f}\frac{\partial t}{\partial \mathbf{b}} + \mathbf{c}\frac{\partial t}{\partial \mathbf{f}} = 0,$$

$$e_4(t) = h\frac{\partial t}{\partial g} + b\frac{\partial t}{\partial f} + 2f\frac{\partial t}{\partial c} + \mathbf{h}\frac{\partial t}{\partial \mathbf{g}} + \mathbf{b}\frac{\partial t}{\partial \mathbf{f}} + 2\mathbf{f}\frac{\partial t}{\partial \mathbf{c}} = 0.$$

The third of the former six equations gives, merely by processes of differentiation, the succession of coefficients for ascending powers of $\phi_{010}$; and the fifth of them gives the succession of coefficients for ascending powers of $\phi_{001}$. The third and the fifth combined give all the coefficients when $t$ is known. When this determination is completed, the seventh and the eighth equations are satisfied identically; and the ninth is satisfied in association with the proper value of $\mu$.

It thus is necessary to consider the above set of four equations. When we associate the equation

$$e_5(t) = 2b\frac{\partial t}{\partial b} - 2c\frac{\partial t}{\partial c} + h\frac{\partial t}{\partial h} - g\frac{\partial t}{\partial g} + 2\mathbf{b}\frac{\partial t}{\partial \mathbf{b}} - 2\mathbf{c}\frac{\partial t}{\partial \mathbf{c}} + \mathbf{h}\frac{\partial t}{\partial \mathbf{h}} - \mathbf{g}\frac{\partial t}{\partial \mathbf{g}} = 0,$$

which is a condition that $e_3(t) = 0$ and $e_4(t) = 0$ should possess solutions in common, the increased set of five equations is found to be a complete Jacobian system. It involves 12 variables, viz., $a$, $b$, $c$, $f$, $g$, $h$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, and therefore it possesses 7 simultaneous independent solutions.

Two of the seven are known; they are

$$\Delta_1 = \mathrm{L}^2 = abc + 2fgh - af^2 - bg^2 - ch^2,$$
$$A = bc = f^2,$$

where $A$ is the "leading" coefficient of the contravariant $\Theta_1$; two others, bearing to the second quadratic the same relation as $\Delta_1$ and $A$ bear to the first, may be expected in the forms

$$\Delta_2 = \mathbf{abc} + 2\mathbf{fgh} - \mathbf{af}^2 - \mathbf{bg}^2 - \mathbf{ch}^2,$$
$$\mathbf{A} = \mathbf{bc} - \mathbf{f}^2.$$

The whole set of seven, including these four, may be constructed by the known processes of solving complete Jacobian systems of homogeneous linear equations.

*Invariants of two Surfaces.*

19. Hitherto we have considered the differential invariants which may arise in connection with a single function $\phi$ of the independent variables; and they may be regarded as associated with the single surface $\phi =$ constant (or with the single family of surfaces if the constant be parametric). But in ordinary space, differential invariants may arise in connection with tortuous curves or (what is the same thing analytically) with two families of surfaces such as $\phi =$ constant, $\phi' =$ constant. They may even arise in connection with three families of surfaces such as $\phi =$ constant, $\phi' =$ constant, $\phi'' =$ constant, where there is no identical functional relation between $\phi$, $\phi'$, $\phi''$ involving constants only.

The simplest of such cases occurs when those differential invariants are required which involve no derivatives of $a$, $b$, $c$, $f$, $g$, $h$ and only first derivatives of two functions $\phi$ and $\phi'$. The number of algebraically independent relative invariants is 4; and the coefficients of the highest power of $\phi_{100}$ which they contain are the independent solutions of the complete Jacobian system

$$2h\frac{\partial}{\partial a} + b\frac{\partial}{\partial h} + f\frac{\partial}{\partial g} + \phi'_{010}\frac{\partial}{\partial \phi'_{100}} = 0,$$

$$2g\frac{\partial}{\partial a} + c\frac{\partial}{\partial g} + f\frac{\partial}{\partial h} + \phi'_{001}\frac{\partial}{\partial \phi'_{100}} = 0,$$

$$2f\frac{\partial}{\partial b} + g\frac{\partial}{\partial h} + c\frac{\partial}{\partial f} + \phi'_{001}\frac{\partial}{\partial \phi'_{010}} = 0,$$

$$2f\frac{\partial}{\partial c} + h\frac{\partial}{\partial g} + b\frac{\partial}{\partial f} + \phi'_{010}\frac{\partial}{\partial \phi'_{001}} = 0,$$

$$2b\frac{\partial}{\partial b} - 2c\frac{\partial}{\partial c} + h\frac{\partial}{\partial h} - g\frac{\partial}{\partial g} + \phi'_{010}\frac{\partial}{\partial \phi'_{010}} - \phi'_{001}\frac{\partial}{\partial \phi'_{001}} = 0.$$

Of these 4, two are already given by $\Delta_1$ and $\Theta_1$. Let

$$2D = \phi'_{100}\frac{\partial}{\partial\phi_{100}} + \phi'_{010}\frac{\partial}{\partial\phi_{010}} + \phi'_{001}\frac{\partial}{\partial\phi_{001}};$$

then other two are given by $D\Theta_1$ and $D^2\Theta_1$. The four quantities

$$\Delta_1, \quad \Theta_1, \quad D\Theta_1, \quad D^2\Theta_1$$

are algebraically independent of one another; and so they constitute the required aggregate of relative differential invariants.

Similarly, we can obtain an algebraically independent aggregate of differential invariants, which involve $a$, $b$, $c$, $f$, $g$, $h$ but none of their derivatives, and which involve first derivatives of three quantities $\phi$, $\phi'$, $\phi''$ unconnected by any identical functional relation. Let

$$2D' = \phi''_{100}\frac{\partial}{\partial\phi_{100}} + \phi''_{010}\frac{\partial}{\partial\phi_{010}} + \phi''_{001}\frac{\partial}{\partial\phi_{001}};$$

then we have, as simultaneous invariants,

$$\Delta_1, \quad \Theta_1, \quad D\Theta_1, \quad D^2\Theta_1, \quad D'\Theta_1, \quad DD'\Theta_1, \quad D'^2\Theta_1,$$

and

$$I = \begin{vmatrix} \phi_{100}, & \phi'_{100}, & \phi''_{100} \\ \phi_{010}, & \phi'_{010}, & \phi''_{010} \\ \phi_{001}, & \phi'_{001}, & \phi''_{001} \end{vmatrix}.$$

There should be only seven invariants in the algebraically complete aggregate; it is not difficult to prove that

$$\begin{vmatrix} \Theta_1, & D\Theta_1, & D'\Theta_1 \\ D\Theta_1, & D^2\Theta_1, & DD'\Theta_1 \\ D'\Theta_1, & DD'\Theta_1, & D'^2\Theta_1 \end{vmatrix} = \Delta_1^3 I^2,$$

so that the above eight quantities, subject to this one relation, constitute the aggregate.

20. We proceed similarly with the determination of the invariants of higher orders associated with two surfaces: here, we shall restrict ourselves to the consideration of those invariants which involve derivatives of $a$, $b$, $c$, $f$, $g$, $h$ of the first order and also involve derivatives of two quantities $\phi$ and $\phi'$ of order not higher than the second. To obtain the expressions of such differential invariants, we take six new quantities

$$\left.\begin{aligned} \mathfrak{a}' &= 2L^2\phi'_{200} - P'a_{100} - Q'(2h_{100} - a_{010}) - R'(2g_{100} - a_{001}) \\ \mathfrak{b}' &= 2L^2\phi'_{020} - P'(2h_{010} - b_{100}) - Q'b_{010} - R'(2f_{010} - b_{001}) \\ \mathfrak{c}' &= 2L^2\phi'_{002} - P'(2g_{001} - c_{100}) - Q'(2f_{001} - c_{010}) - R'c_{001} \\ \mathfrak{f}' &= 2L^2\phi'_{011} - P'(-f_{100} + g_{010} + h_{001}) - Q'b_{001} - R'c_{010} \\ \mathfrak{g}' &= 2L^2\phi'_{101} - P'a_{001} - Q'(f_{100} - g_{010} + h_{001}) - R'c_{100} \\ \mathfrak{h}' &= 2L^2\phi'_{110} - P'a_{010} - Q'b_{100} - R'(f_{100} + g_{010} - h_{001}) \end{aligned}\right\},$$

where

$$\left.\begin{aligned} P' &= A\phi'_{100} + H\phi'_{010} + G\phi'_{001} \\ Q' &= H\phi'_{100} + B\phi'_{010} + F\phi'_{001} \\ R' &= G\phi'_{100} + F\phi'_{010} + C\phi'_{001} \end{aligned}\right\}.$$

These quantities bear to $\phi'$ the same relation as the quantities $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ bear to $\phi$.

The differential invariants up to the order specified are the simultaneous invariants and contravariants of the simultaneous ternary quantics

$$(\phi'_{100}, \phi'_{010}, \phi'_{001} \between X, Y, Z),$$
$$(a, b, c, f, g, h \between X, Y, Z)^2,$$
$$(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h} \between X, Y, Z)^2,$$
$$(\mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{f}', \mathfrak{g}', \mathfrak{h}' \between X, Y, Z)^2,$$

the variables of the contravariants being $\phi_{100}$, $\phi_{010}$, $\phi_{001}$. The total number of algebraically independent relative differential invariants is 16, being the number of independent solutions of five equations (as in § 18) which are a complete Jacobian set and involve 21 variables. Hence the total number of algebraically independent absolute differential invariants of the specified order involving derivatives of two functions $\phi$ and $\phi'$ is 15.

As regards the indices of the respective relative invariants, they are given as before. Let the invariant be homogeneous in $\phi_{100}$, $\phi_{010}$, $\phi_{001}$ of degree $n$; in $\phi'_{100}$, $\phi'_{010}$, $\phi'_{001}$ of degree $n'$; in $a, b, c, f, g, h$ of degree $m$; in $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$ of degree $l$; and in $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{f}'$, $\mathfrak{g}'$, $\mathfrak{h}'$ of degree $l'$. Then its index $\mu$ is given by

$$8(l + l') + 2m + n + n' = 3\mu.$$

21. The algebraically complete aggregate of sixteen relative invariants can be expressed in various forms that are equivalent to one another. One such aggregate of invariants of the second order can be obtained as follows. We write

$$2D = \phi'_{100} \frac{\partial}{\partial \phi_{100}} + \phi'_{010} \frac{\partial}{\partial \phi_{010}} + \phi'_{001} \frac{\partial}{\partial \phi_{001}};$$

and we use the results of § 16.

If there were only one surface $\phi$, the aggregate of invariants of the second order would be

$$\Delta_1, \quad \Delta_{12}, \quad \Delta_{21}, \quad \Delta_2, \quad \Theta_1, \quad \Theta_{12}, \quad \Theta_2.$$

As there is a second surface $\phi'$, introducing a third ternary quadratic, the aggregate of invariants of the second order must include concomitants that may be denoted by

$$\Delta_{13}, \quad \Delta_{31}, \quad \Delta_3, \quad \Theta_{13}, \quad \Theta_3$$

(or their equivalent), these concomitants having the same relation to the third quadratic as $\Delta_{12}$, $\Delta_{21}$, $\Delta_2$, $\Theta_{12}$, $\Theta_2$ to the second quadratic. Moreover, in the case of two surfaces $\phi$ and $\phi'$, the aggregate of invariants of the first order is

$$\Delta_1, \quad \Theta_1, \quad D\Theta_1, \quad D^2\Theta_1;$$

and these must be included in the desired aggregate. Further, as regards the concomitants, the second and the third quadratics will have invariants, standing to them in the same relations as $\Delta_{12}$ and $\Delta_{21}$ to the first and the second or as $\Delta_{13}$ and $\Delta_{31}$ to the first and the third; we therefore have $\Delta_{23}$ and $\Delta_{32}$ as invariants.

Accordingly, we take the algebraic aggregate of sixteen relative invariants as composed of the quantities

$$\begin{array}{lllll} \Delta_1, & \Theta_1, & & & \\ \Delta_{12}, & \Delta_{21}, & \Delta_2, & \Theta_{12}, & \Theta_2, \\ \Delta_{13}, & \Delta_{31}, & \Delta_3, & \Theta_{13}, & \Theta_3, \\ D\Theta_1, & D^2\Theta_1, & & & \\ \Delta_{23}, & \Delta_{32}. & & & \end{array}$$

Let $2\mathfrak{C}$ denote the operator $\phi_{100}\dfrac{\partial}{\partial\phi'_{100}} + \phi_{010}\dfrac{\partial}{\partial\phi'_{010}} + \phi_{001}\dfrac{\partial}{\partial\phi'_{001}}$; then other relative invariants are given by

$$\begin{array}{llllll} D\Theta_2, & D^2\Theta_2, & D\Theta_{12}, & D^2\Theta_{12}, & D\Theta_{13}, & D^2\Theta_{13}, \\ \mathfrak{C}\Theta_3, & \mathfrak{C}^2\Theta_3, & & & & \end{array}$$

and so on; all of these are algebraically expressible in terms of the above aggregate of sixteen.

The umbral expressions for these invariants are easily obtained. We take the notation of § 17, and we further introduce a third set of umbral symbols $a''_1$, $a''_2$, $a''_3$; $b''_1$, $b''_2$, $b''_3$; and so on, defined as

$$(\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{f}', \mathrm{g}', \mathrm{h}' \between X, Y, Z)^2 = a''^2_x = b''^2_x = \ldots$$

We also write

$$\phi'_{100} = u'_1, \quad \phi'_{010} = u'_2, \quad \phi'_{001} = u'_3;$$

and then we have the following expressions for the sixteen relative invariants:—

$$\begin{array}{lll} \Delta_1 = \frac{1}{6}(abc)^2, & \Delta_{12} = \frac{1}{2}(a'ab)^2, & \Delta_{13} = \frac{1}{2}(a''ab)^2, \\ \Delta_2 = \frac{1}{6}(a'b'c')^2, & \Delta_{21} = \frac{1}{2}(aa'b')^2, & \Delta_{23} = \frac{1}{2}(a''a'b')^2, \\ \Delta_3 = \frac{1}{6}(a''b''c'')^2, & \Delta_{31} = \frac{1}{2}(aa''b'')^2, & \Delta_{32} = \frac{1}{2}(a'a''b'')^2, \\ \Theta_1 = \frac{1}{2}(abu)^2, & \Theta_{12} = (aa'u)^2, & \\ \Theta_2 = \frac{1}{2}(a'b'u)^2, & \Theta_{13} = (aa''u')^2, & \\ \Theta_3 = \frac{1}{2}(a''b''u')^2, & & \end{array}$$

$$D\Theta_1 = \frac{1}{2}(abu)(abu'), \qquad D^2\Theta_1 = \frac{1}{2}(abu')^2.$$

Other invariants manifestly are given by

$$(aa'a'')^2, \quad \tfrac{1}{2}(a'b'u)(a'b'u'), \quad \tfrac{1}{2}(a'b'u')^2, \quad \tfrac{1}{2}(a''b''u)(a''b''u'), \quad \tfrac{1}{2}(a''b''u)^2,$$

and so on; all of them are algebraically expressible in terms of the above aggregate of sixteen.

The values of the indices of the respective invariants are given by the relation

$$8(l + l') + 2m + n + n' = 3\mu$$

in each case. The results are as follows :—

| | | | | |
|---|---|---|---|---|
| Index $= 2$, | $\Delta_1$, | $\Theta_1$, | $D\Theta_1$, | $D^2\Theta_1$, |
| . . $= 4$, | $\Delta_{12}$, | $\Delta_{13}$, | $\Theta_{12}$, | $\Theta_{13}$, |
| . . $= 6$, | $\Delta_{21}$, | $\Delta_{31}$, | $\Theta_2$, | $\Theta_3$, |
| . . $= 8$, | $\Delta_2$, | $\Delta_3$, | $\Delta_{23}$, | $\Delta_{32}$. |

The expressions of the 15 absolute invariants now are obvious.

## *Invariants of the Third Order.*

22. The calculations, involved in this mode of constructing differential invariants, are very laborious for differential invariants of the third order, being the order next in succession. They are so extensive and demand such sustained attention merely through long processes of algebra that, if invariants of higher order are required, it will (in my opinion) be necessary to devise some other method of construction.

Only the briefest outline of what has been done in the case of differential invariants of the third order will be given, so far as concerns these laborious processes; the results will be stated.

23. The invariants in question involve derivatives of $\phi$ up to the third order inclusive; they also involve the quantities $a$, $b$, $c$, $f$, $g$, $h$, as well as their derivatives up to the second order inclusive.

The differential equations characteristic of the invariants are 57 in number; and they range themselves in three sets. The first set consists of the thirty equations, which arise from the derivatives of $\xi$, $\eta$, $\zeta$ of the third order in the course of the process indicated in § 7; the second set consists of the eighteen equations which similarly arise from the derivatives of $\xi$, $\eta$, $\zeta$ of the second order; and the third set consists of the nine equations which similarly arise from the derivatives of $\xi$, $\eta$, $\zeta$ of the first order.

The actual formation of the differential equations is effected as in § 7. All that is needed for the purpose, in addition to the results in § 6, is the aggregate of the expressions of the increments of the second derivatives of $a$, $b$, $c$, $f$, $g$, $h$ and of the third derivatives of $\phi$; and these are special instances of the formulæ in § 5.

The first set of equations, consisting of 30 members, possesses sixteen independent integrals, as well as $a, b, c, f, g, h$, their derivatives of the first order, and the derivatives of $\phi$ of the second order. The sixteen integrals are :—

$$\theta_1 = b_{002} + c_{020} - 2f_{011},$$

$$\theta_2 = c_{200} + a_{002} - 2g_{101},$$

$$\theta_3 = a_{020} + b_{200} - 2h_{110},$$

$$\theta_4 = a_{011} - g_{110} - h_{101} + f_{200},$$

$$\theta_5 = b_{101} - h_{011} - f_{110} + g_{020},$$

$$\theta_6 = c_{110} - f_{101} - g_{011} + h_{002};$$

$$u_1 = 2L^2\phi_{300} - Pa_{200} - Q(2h_{200} - a_{110}) - R(2g_{200} - a_{101}),$$

$$u_2 = 2L^2\phi_{210} - Pa_{110} - Qb_{200} - R(2g_{110} - a_{011}),$$

$$u_3 = 2L^2\phi_{201} - Pa_{101} - Q(2h_{101} - a_{011}) - Rc_{200},$$

$$u_4 = 2L^2\phi_{120} - Pa_{020} - Qb_{110} - R(2f_{110} - b_{101}),$$

$$u_5 = 2L^2\phi_{111} - Pa_{011} - Qb_{101} - Rc_{110},$$

$$u_6 = 2L^2\phi_{102} - Pa_{002} - Q(2f_{101} - c_{110}) - Rc_{101},$$

$$u_7 = 2L^2\phi_{030} - P(2h_{020} - b_{110}) - Qb_{020} - R(2f_{020} - b_{011}),$$

$$u_8 = 2L^2\phi_{021} - P(2h_{011} - b_{101}) - Qb_{011} - Rc_{020},$$

$$u_9 = 2L^2\phi_{012} - P(2g_{011} - c_{110}) - Qb_{002} - Rc_{011},$$

$$u_{10} = 2L^2\phi_{003} - P(2g_{002} - c_{101}) - Q(2f_{002} - c_{011}) - Rc_{002}.$$

All these quantities are integrals of the 30 equations, as also are all functional combinations of them. The integrals of the remaining equations must accordingly be some functional combinations of $\theta_1, \ldots, \theta_6, u_1, \ldots, u_{10}$, as well as of $a, b, c, f, g, h$, their derivatives of the first order, and the derivatives of $\phi$ of the second order.

The second set of equations, consisting of 18 members, possesses sixteen independent integrals, in addition to $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h}$, $a, b, c, f, g, h$, and the derivatives of $\phi$ of the first order. The sixteen integrals are :—

$$\Theta_1 = 2L^2\theta_1 + A\alpha'_1 + H\beta'_1 + G\gamma'_1 + B\delta'_1 + F\epsilon'_1 + C\zeta'_1$$

$$\Theta_2 = 2L^2\theta_2 + A\alpha'_2 + H\beta'_2 + G\gamma'_2 + B\delta'_2 + F\epsilon'_2 + C\zeta'_2,$$

$$\Theta_3 = 2L^2\theta_3 + A\alpha'_3 + H\beta'_3 + G\gamma'_3 + B\delta'_3 + F\epsilon'_3 + C\zeta'_3,$$

$$\Theta_4 = 2L^2\theta_4 + A\alpha'_4 + H\beta'_4 + G\gamma'_4 + B\delta'_4 + F\epsilon'_4 + C\zeta'_4,$$

$$\Theta_5 = 2L^2\theta_5 + A\alpha'_5 + H\beta'_5 + G\gamma'_5 + B\delta'_5 + F\epsilon'_5 + C\zeta'_5,$$

$$\Theta_6 = 2L^2\theta_6 + A\alpha'_6 + H\beta'_6 + G\gamma'_6 + B\delta'_6 + F\epsilon'_6 + C\zeta'_6\,;$$

$$\mathfrak{a}'' = 2L^2u_1 - \alpha_1\phi_{200} - \beta_1\phi_{110} - \gamma_1\phi_{101} - P\lambda_1 - Q\mu_1 - R\nu_1,$$

$$\mathfrak{k}'' = 2L^2u_7 - \alpha_7\phi_{020} - \beta_7\phi_{110} - \gamma_7\phi_{011} - P\lambda_7 - Q\mu_7 - R\nu_7,$$

$$\mathfrak{n}'' = 2L^2u_{10} - \alpha_{10}\phi_{002} - \beta_{10}\phi_{011} - \gamma_{10}\phi_{101} - P\lambda_{10} - Q\mu_{10} - R\nu_{10},$$

$$\mathfrak{h}'' = 2L^2u_2 - \alpha_2\phi_{200} - \beta_2\phi_{110} - \gamma_2\phi_{101} - \delta_2\phi_{020} - \epsilon_2\phi_{011} - P\lambda_2 - Q\mu_2 - R\nu_2,$$

$$\mathfrak{g}'' = 2L^2u_3 - \alpha_3\phi_{200} - \beta_3\phi_{110} - \gamma_3\phi_{101} - \epsilon_3\phi_{011} - \zeta_3\phi_{002} - P\lambda_3 - Q\mu_3 - R\nu_3,$$

$$\mathfrak{b}'' = 2L^2u_4 - \alpha_4\phi_{200} - \beta_4\phi_{110} - \gamma_4\phi_{101} - \delta_4\phi_{020} - \epsilon_4\phi_{011} - P\lambda_4 - Q\mu_4 - R\nu_4,$$

$$\mathfrak{c}'' = 2L^2u_6 - \alpha_6\phi_{200} - \beta_6\phi_{110} - \gamma_6\phi_{101} - \epsilon_6\phi_{011} - \zeta_6\phi_{002} - P\lambda_6 - Q\mu_6 - R\nu_6,$$

$$\mathfrak{l}'' = 2L^2u_8 - \beta_8\phi_{110} - \gamma_8\phi_{101} - \delta_8\phi_{020} - \epsilon_8\phi_{011} - \zeta_8\phi_{002} - P\lambda_8 - Q\mu_8 - R\nu_8,$$

$$\mathfrak{m}'' = 2L^2u_9 - \beta_9\phi_{110} - \gamma_9\phi_{101} - \delta_9\phi_{020} - \epsilon_9\phi_{011} - \zeta_9\phi_{002} - P\lambda_9 - Q\mu_9 - R\nu_9,$$

$$\mathfrak{f}'' = 2L^2u_5 - \alpha_5\phi_{200} - \beta_5\phi_{110} - \gamma_5\phi_{101} - \delta_5\phi_{020} - \epsilon_5\phi_{011} - \zeta_5\phi_{002} - P\lambda_5 - Q\mu_5 - R\nu_5.$$

The various coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$, $\lambda$, $\mu$, $\nu$ are independent of the derivatives of $\phi$.

The coefficients in the six integrals $\Theta_1$, $\Theta_2$, $\Theta_3$, $\Theta_4$, $\Theta_5$ $\Theta_6$ are as follows:—

$$\left.\begin{aligned}
\alpha'_1 &= b_{100}c_{100} - 2b_{100}g_{001} - 2c_{100}h_{010} + 4g_{001}h_{010} - (-f_{100} + g_{010} + h_{001})^2, \\
\beta'_1 &= b_{100}c_{010} - 2b_{100}f_{001} + 2b_{010}g_{001} - b_{010}c_{100} + 2b_{001}f_{100} - 2b_{001}g_{010} - 2b_{001}h_{001} + 4f_{001}h_{010} - 2c_{010}h_{010}, \\
\gamma'_1 &= b_{001}c_{100} - 2c_{100}f_{010} + 2c_{001}h_{010} - b_{100}c_{001} + 2c_{010}f_{100} - 2c_{010}h_{001} - 2c_{010}g_{010} + 4f_{010}g_{001} - 2b_{001}g_{001}, \\
\delta'_1 &= -b^2_{001} + 2b_{010}f_{001} - b_{010}c_{010}, \\
\epsilon'_1 &= b_{010}c_{001} - b_{001}c_{010} - 2b_{001}f_{001} - 2c_{010}f_{010} + 4f_{010}f_{001}, \\
\zeta'_1 &= -c^2_{010} + 2c_{001}f_{010} - b_{001}c_{001}\,;
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\alpha'_2 &= -a^2_{001} + 2a_{100}g_{001} - a_{100}c_{100}, \\
\beta'_2 &= a_{010}c_{100} - 2a_{010}g_{001} + 2a_{100}f_{001} - a_{100}c_{010} + 2a_{001}g_{010} - 2a_{001}f_{100} - 2a_{001}h_{001} + 4g_{001}h_{100} - 2c_{100}h_{100}, \\
\gamma'_2 &= a_{100}c_{001} - a_{001}c_{100} - 2a_{001}g_{001} - 2c_{100}g_{100} + 4g_{100}g_{001}, \\
\delta'_2 &= a_{010}c_{010} - 2a_{010}f_{001} - 2c_{010}h_{100} + 4f_{001}h_{100} - (f_{100} - g_{010} + h_{001})^2, \\
\epsilon'_2 &= a_{001}c_{010} - 2c_{010}g_{100} + 2c_{001}h_{100} - a_{010}c_{001} + 2c_{100}g_{010} - 2c_{100}h_{001} - 2c_{100}f_{100} + 4f_{001}g_{100} - 2a_{001}f_{001}, \\
\zeta'_2 &= -c^2_{100} + 2c_{001}g_{100} - a_{001}c_{001}\,;
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\alpha'_3 &= -a^2_{010} + 2a_{100}h_{010} - a_{100}b_{100}, \\
\beta'_3 &= a_{100}b_{010} - a_{010}b_{100} - 2b_{100}h_{100} - 2a_{010}h_{010} + 4h_{100}h_{010}, \\
\gamma'_3 &= a_{001}b_{100} - 2a_{001}h_{010} + 2a_{100}f_{010} - a_{100}b_{001} + 2a_{010}h_{001} - 2a_{010}f_{100} - 2a_{010}g_{010} + 4g_{100}h_{010} - 2b_{100}g_{100}, \\
\delta'_3 &= -b^2_{100} + 2b_{010}h_{100} - a_{010}b_{010}, \\
\epsilon'_3 &= a_{010}b_{001} - 2b_{001}h_{100} + 2b_{010}g_{100} - a_{001}b_{010} + 2b_{100}h_{001} - 2b_{100}g_{010} - 2b_{100}f_{100} + 4f_{010}h_{100} - 2a_{010}f_{010}, \\
\zeta'_3 &= a_{001}b_{001} - 2b_{001}g_{100} - 2a_{001}f_{010} + 4g_{100}f_{010} - (f_{100} + g_{010} - h_{001})^2\,;
\end{aligned}\right\}$$

$$\left\{\begin{aligned}
\alpha'_4 &= -a_{100}f_{100} + a_{100}g_{010} + a_{100}h_{001} - a_{010}a_{001},\\
\beta'_4 &= a_{100}b_{001} - 2a_{010}h_{001} - a_{001}b_{100} - 2f_{100}h_{100} + 2g_{010}h_{100} + 2h_{100}h_{001},\\
\gamma'_4 &= a_{100}c_{010} - 2a_{001}g_{010} - a_{010}c_{100} - 2f_{100}g_{100} + 2g_{100}h_{001} + 2g_{100}g_{010},\\
\delta'_4 &= -a_{010}b_{001} - b_{100}f_{100} + b_{100}g_{010} - b_{100}h_{001} + 2b_{001}h_{100},\\
\epsilon'_4 &= -a_{010}c_{010} - a_{001}b_{001} - b_{100}c_{100} + 2b_{001}g_{100} + 2c_{010}h_{100} - f^2_{100} + g^2_{010} - 2g_{010}h_{001} + h^2_{001},\\
\zeta'_4 &= -a_{001}c_{010} - c_{100}f_{100} + c_{100}h_{001} - c_{100}g_{010} + 2c_{010}g_{100};
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\alpha'_5 &= -a_{001}b_{100} - a_{010}g_{010} + a_{010}f_{100} - a_{010}h_{001} + 2a_{001}h_{010},\\
\beta'_5 &= a_{001}b_{010} - 2b_{100}h_{001} - a_{010}b_{001} - 2g_{010}h_{010} + 2f_{100}h_{010} + 2h_{010}h_{001},\\
\gamma'_5 &= -b_{100}c_{100} - a_{001}b_{001} - a_{010}c_{010} + 2a_{001}f_{010} + 2c_{100}h_{010} + f^2_{100} - g^2_{010} - 2f_{100}h_{001} + h^2_{001},\\
\delta'_5 &= -b_{010}g_{010} + b_{010}f_{100} + b_{010}h_{001} - b_{100}b_{001},\\
\epsilon'_5 &= b_{010}c_{100} - 2b_{001}f_{100} - b_{100}c_{010} - 2f_{010}g_{010} + 2f_{010}h_{001} + 2f_{100}f_{010},\\
\zeta'_5 &= -b_{001}c_{100} - c_{010}g_{010} + c_{010}h_{001} - c_{010}f_{100} + 2c_{100}f_{010};
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\alpha'_6 &= -a_{010}c_{100} - a_{001}h_{001} + a_{001}f_{100} - a_{001}g_{010} + 2a_{010}g_{001},\\
\beta'_6 &= -a_{010}c_{010} - b_{100}c_{100} - a_{001}b_{001} + 2b_{100}g_{001} + 2a_{010}f_{001} + f^2_{100} + g^2_{010} - 2f_{100}g_{010} - h^2_{001},\\
\gamma'_6 &= a_{010}c_{001} - 2c_{100}g_{010} - a_{001}c_{010} - 2g_{001}h_{001} + 2f_{100}g_{001} + 2g_{010}g_{001},\\
\delta'_6 &= -b_{100}c_{010} - b_{001}h_{001} + b_{001}g_{010} - b_{001}f_{100} + 2b_{100}f_{001},\\
\epsilon'_6 &= b_{100}c_{001} - 2c_{010}f_{100} - b_{001}c_{100} - 2f_{001}h_{001} + 2f_{001}g_{010} + 2f_{100}f_{001},\\
\zeta'_6 &= -c_{001}h_{001} + c_{001}g_{010} + c_{001}f_{100} - c_{100}c_{010}.
\end{aligned}\right.$$

The coefficients in the ten integrals $\mathfrak{a}''$, $\mathfrak{b}''$, $\mathfrak{c}''$, $\mathfrak{f}''$, $\mathfrak{g}''$, $\mathfrak{h}''$, $\mathfrak{k}''$, $\mathfrak{l}''$, $\mathfrak{m}''$, $\mathfrak{n}''$ are as follows :—

$$\alpha_1 = 6\mathrm{L}^2\{\mathrm{A}a_{100} + \mathrm{H}(2h_{100} - a_{010}) + \mathrm{G}(2g_{100} - a_{001})\},$$

$$\beta_1 = 6\mathrm{L}^2\{\mathrm{H}a_{100} + \mathrm{B}(2h_{100} - a_{010}) + \mathrm{F}(2g_{100} - a_{001})\},$$

$$\gamma_1 = 6\mathrm{L}^2\{\mathrm{G}a_{100} + \mathrm{F}(2h_{100} - a_{010}) + \mathrm{C}(2g_{100} - a_{001})\},$$

$$\begin{aligned}
\lambda_1 = {} & -4\mathrm{A}a^2_{100} + 2\mathrm{B}(a^2_{010} - a_{010}h_{100} - 2h^2_{100}) + 2\mathrm{C}(a^2_{001} - a_{001}g_{100} - 2g^2_{100})\\
& + 2\mathrm{F}(2a_{010}a_{001} - a_{010}g_{100} - a_{001}h_{100} - 4g_{100}h_{100})\\
& + 2\mathrm{G}(a_{100}a_{001} - 5a_{100}g_{100}) + 2\mathrm{H}(a_{100}a_{010} - 5a_{100}h_{100}),
\end{aligned}$$

$$\begin{aligned}
\mu_1 = {} & 2\mathrm{A}(a_{100}a_{010} - 3a_{100}h_{100}) + 2\mathrm{B}(2a_{010}b_{100} - 4b_{100}h_{100})\\
& + 2\mathrm{C}(2a_{001}f_{100} - a_{001}g_{010} + a_{001}h_{001} - 4f_{100}g_{100} + 2g_{010}g_{100} - 2g_{100}h_{001})\\
& + 2\mathrm{F}(2a_{010}f_{100} - a_{010}g_{010} + a_{010}h_{001} + 2a_{001}b_{100}\\
& \qquad - 4b_{100}g_{100} - 4f_{100}h_{100} + 2g_{010}h_{100} - 2h_{001}h_{100})\\
& + 2\mathrm{G}(-2a_{100}f_{100} + a_{100}g_{010} - a_{100}h_{001} + 2a_{010}g_{100} - a_{010}a_{001} + 3h_{100}a_{001} - 6g_{100}h_{100})\\
& + 2\mathrm{H}(-2a_{100}b_{100} + 5a_{010}h_{100} - a^2_{010} - 6h^2_{100}),
\end{aligned}$$

$$\begin{aligned}
\nu_1 = {} & 2\mathrm{A}(a_{100}a_{001} - 3a_{100}g_{100}) + 2\mathrm{C}(2a_{001}c_{100} - 4c_{100}g_{100})\\
& + 2\mathrm{B}(2a_{010}f_{100} - a_{010}h_{001} + a_{010}g_{010} - 4f_{100}h_{100} + 2h_{100}h_{001} - 2g_{010}h_{100})\\
& + 2\mathrm{F}(2a_{001}f_{100} - a_{001}h_{001} + a_{001}g_{010} + 2a_{010}c_{100}\\
& \qquad - 4c_{100}h_{100} - 4f_{100}g_{100} + 2g_{100}h_{001} - 2g_{100}g_{010})\\
& + 2\mathrm{G}(-2a_{100}c_{100} + 5a_{001}g_{100} - a^2_{001} - 6g^2_{100})\\
& + 2\mathrm{H}(-2a_{100}f_{100} + a_{100}h_{001} - a_{100}g_{010} + 2a_{001}h_{100} - a_{010}a_{001} + 3g_{100}a_{010} - 6g_{100}h_{100});
\end{aligned}$$

$$\alpha_2 = 4L^2 \{Aa_{010} + Hb_{100} + G(f_{100} + g_{010} - h_{001})\},$$

$$\beta_2 = 2L^2 \{Aa_{100} + H(2h_{100} + a_{010}) + G(2g_{100} - a_{001}) + 2Bb_{100} + 2F(f_{100} + g_{010} - h_{001})\},$$

$$\gamma_2 = 4L^2 \{Ga_{010} + Fb_{100} + C(f_{100} + g_{010} - h_{001})\},$$

$$\delta_2 = 2L^2 \{Ha_{100} + B(2h_{100} - a_{010}) + F(2g_{100} - a_{001})\},$$

$$\epsilon_2 = 2L^2 \{Ga_{100} + F(2h_{100} - a_{010}) + C(2g_{100} - a_{001})\},$$

$$\begin{aligned}\lambda_2 = & - 4Aa_{100}a_{010} - 2H(a_{100}b_{100} + a_{100}h_{010} + 2a_{010}h_{100}) + 2G(- a_{100}f_{100} - 2a_{100}g_{010} + a_{100}h_{001} - 2a_{010}g_{100}) \\ & + 2B(a_{010}h_{010} - a_{010}b_{100} - 2h_{010}h_{100}) + 2F(a_{001}h_{010} - a_{001}b_{100} + a_{010}h_{001} - a_{010}f_{100} - 2g_{100}h_{010} - 2g_{010}h_{100}) \\ & + 2C(- a_{001}f_{100} + a_{001}h_{001} - 2g_{100}g_{010}),\end{aligned}$$

$$\begin{aligned}\mu_2 = & \, A(a^2_{010} - 4a_{010}h_{100} - a_{100}b_{100}) + B(a_{010}b_{010} - 3b^2_{100} - 2b_{010}h_{100}) \\ & + C(a_{001}b_{001} - 2b_{001}g_{100} - 2f_{100}g_{010} + 2f_{100}h_{001} - 2g_{010}h_{001} - 3f^2_{100} + g^2_{010} + h^2_{001}) \\ & + F(a_{010}b_{001} + a_{001}b_{010} - 6b_{100}f_{100} - 2b_{100}g_{010} + 2b_{100}h_{001} - 2b_{010}g_{100} - 2b_{001}h_{100}) + H(- a_{100}b_{010} - a_{010}b_{100} - 6b_{100}h_{100}) \\ & + G(- a_{100}b_{001} - 2a_{010}f_{100} + 2a_{010}g_{010} - 2a_{010}h_{001} + a_{001}b_{100} - 2b_{100}g_{100} - 4f_{100}h_{100} - 4g_{010}h_{100} + 4h_{100}h_{001}),\end{aligned}$$

$$\begin{aligned}\nu_2 = & \, 2A(- a_{100}g_{010} - 2a_{010}g_{100} + a_{010}a_{001}) + 2B(- 2f_{010}h_{100} + a_{010}f_{010} - b_{100}f_{100} - b_{100}g_{010} + b_{100}h_{001}) \\ & + 2C(a_{001}c_{010} - c_{100}f_{100} - c_{100}g_{010} + c_{100}h_{001} - 2c_{010}g_{100}) \\ & + 2H(- a_{100}f_{010} - 2g_{010}h_{100} - a_{010}f_{100} + a_{010}h_{001} + a_{001}b_{100} - 2b_{100}g_{100}) \\ & + 2F(- 2c_{010}h_{100} + a_{010}c_{010} + a_{001}f_{010} - b_{100}c_{100} - f^2_{100} - 2f_{100}g_{010} + 2f_{100}h_{001} + 2g_{010}h_{001} - g^2_{010} - h^2_{001} - 2f_{010}g_{100}) \\ & + 2G(- a_{100}c_{010} - a_{010}c_{100} + a_{001}f_{100} + 2a_{001}g_{010} - a_{001}h_{001} - 2g_{100}f_{100} - 4g_{100}g_{010} + 2g_{100}h_{001});\end{aligned}$$

$$\alpha_3 = 4L^2 \{Aa_{001} + H(f_{100} - g_{010} + h_{001}) + Gc_{100}\},$$

$$\beta_3 = 4L^2 \{Ha_{001} + B(f_{100} - g_{010} + h_{001}) + Fc_{100}\},$$

$$\gamma_3 = 2L^2 \{Aa_{100} + H(2h_{100} - a_{010}) + G(2g_{100} + a_{001}) + 2F(f_{100} - g_{010} + h_{001}) + 2Cc_{100}\},$$

$$\epsilon_3 = 2L^2 \{Ha_{100} + B(2h_{100} - a_{010}) + F(2g_{100} - a_{001})\},$$

$$\zeta_3 = 2L^2 \{Ga_{100} + F(2h_{100} - a_{010}) + C(2g_{100} - a_{001})\},$$

$$\begin{aligned}\lambda_3 = &- 4Aa_{100}a_{001} + 2B(- a_{010}f_{100} + a_{010}g_{010} - 2h_{100}h_{001}) + 2C(a_{001}g_{001} - a_{001}c_{100} - 2g_{001}g_{100}) \\ &- 2G(a_{100}c_{100} + a_{100}g_{001} + 2a_{001}g_{100}) + 2F(a_{010}g_{001} - a_{010}c_{100} + a_{001}g_{010} - a_{001}f_{100} - 2g_{001}h_{100} - 2g_{100}h_{001}) \\ &+ 2H(- a_{100}f_{100} - 2a_{100}h_{001} + a_{100}g_{010} - 2a_{001}h_{100}),\end{aligned}$$

$$\begin{aligned}\mu_3 = &\, 2A(-a_{100}h_{001} - 2a_{001}h_{100} + a_{010}a_{001}) + 2B(a_{010}b_{001} - b_{100}f_{100} - b_{100}h_{001} + b_{100}g_{010} - 2b_{001}h_{100}) \\ &+ 2C(- 2f_{001}g_{100} + a_{001}f_{001} - c_{100}f_{100} - c_{100}h_{001} + c_{100}g_{010}) \\ &+ 2G(- a_{100}f_{001} - 2g_{100}h_{001} - a_{001}f_{100} + a_{001}g_{010} + a_{010}c_{100} - 2c_{100}h_{100}) \\ &+ 2F(- 2b_{001}g_{100} + a_{001}b_{001} + a_{010}f_{001} - b_{100}c_{100} - f^2_{100} + 2f_{100}g_{010} - 2f_{100}h_{001} - g^2_{010} + 2g_{010}h_{001} - h^2_{001} - 2f_{001}h_{100}) \\ &+ 2H(- a_{100}b_{001} - a_{001}b_{100} + a_{010}f_{100} + 2a_{010}h_{001} - a_{010}g_{010} - 2f_{100}h_{100} - 4h_{100}h_{001} + 2g_{010}h_{100}),\end{aligned}$$

$$\begin{aligned}\nu_3 = &\, A(a^2_{001} - 4a_{001}g_{100} - a_{100}c_{100}) + C(a_{001}c_{001} - 3c^2_{100} - 2c_{001}g_{010}) \\ &+ B(a_{010}c_{010} - 2c_{010}h_{100} - 2f_{100}h_{001} + 2f_{100}g_{010} - 2g_{010}h_{001} - 3f^2_{100} + g^2_{010} + h^2_{001}) \\ &+ H(- a_{100}c_{010} - 2a_{001}f_{100} + 2a_{001}h_{001} - 2a_{001}g_{010} + a_{010}c_{100} - 2c_{100}h_{100} - 4f_{100}g_{100} - 4g_{100}h_{001} + 4g_{100}g_{010}) \\ &+ F(a_{001}c_{010} + a_{010}c_{001} - 6c_{100}f_{100} - 2c_{100}h_{001} + 2c_{100}g_{010} - 2c_{001}h_{100} - 2c_{010}g_{100}) + G(- a_{100}c_{001} - a_{001}c_{100} - 6c_{100}g_{100});\end{aligned}$$

$$\alpha_4 = 2L^2 \{A(2h_{010} - b_{100}) + Hb_{010} + G(2f_{010} - b_{001})\},$$

$$\beta_4 = 2L^2 \{2Aa_{010} + H(2h_{010} + b_{100}) + 2G(f_{100} + g_{010} - h_{001}) + Bb_{010} + F(2f_{010} - b_{001})\},$$

$$\gamma_4 = 2L^2 \{G(2h_{010} - b_{100}) + Fb_{010} + C(2f_{010} - b_{001})\},$$

$$\delta_4 = 4L^2 \{Ha_{010} + Bb_{100} + F(f_{100} + g_{010} - h_{001})\},$$

$$\epsilon_4 = 4L^2 \{Ga_{010} + Fb_{100} + C(f_{100} + g_{010} - h_{001})\},$$

$$\begin{aligned}\lambda_4 = {} & A(a_{100}b_{100} - 3a^2_{010} - 2a_{100}h_{010}) + B(b^2_{100} - 4b_{100}h_{010} - a_{010}b_{010}) \\ & + C(a_{001}b_{001} - 2a_{001}f_{010} - 2f_{100}g_{010} + 2g_{010}h_{001} - 2f_{100}h_{001} - 3g^2_{010} + f^2_{100} + h^2_{001}) \\ & + F(-a_{001}b_{010} - 2b_{100}g_{010} + 2b_{100}f_{100} - 2b_{100}h_{001} + a_{010}b_{001} - 2a_{010}f_{010} - 4g_{010}h_{010} - 4f_{100}h_{010} + 4h_{010}h_{001}) \\ & + G(a_{001}b_{100} + a_{100}b_{001} - 6a_{010}g_{010} - 2a_{010}f_{100} + 2a_{010}h_{001} - 2a_{100}f_{010} - 2a_{001}h_{010}) + H(-a_{100}b_{010} - a_{010}b_{100} - 6a_{010}h_{010}),\end{aligned}$$

$$\begin{aligned}\mu_4 = {} & 2A(b_{100}h_{100} - a_{010}b_{100} - 2h_{010}h_{100}) - 4Bb_{100}b_{010} + 2C(-b_{001}g_{010} + b_{001}h_{001} - 2f_{100}f_{010}) - 2H(a_{010}b_{010} + b_{010}h_{100} + 2b_{100}h_{010}) \\ & + 2F(-b_{010}g_{010} - 2b_{010}f_{100} + b_{010}h_{001} - 2b_{100}f_{010}) + 2G(b_{001}h_{100} - a_{010}b_{001} + b_{100}h_{001} - b_{100}g_{010} - 2f_{010}h_{100} - 2f_{100}h_{010}),\end{aligned}$$

$$\begin{aligned}\nu_4 = {} & 2A(-2g_{100}h_{010} + b_{100}g_{100} - a_{010}g_{010} - a_{010}f_{100} + a_{010}h_{001}) + 2B(-b_{010}f_{100} - 2b_{100}f_{010} + b_{100}b_{001}) \\ & + 2C(b_{001}c_{100} - c_{010}g_{010} - c_{010}f_{100} + c_{010}h_{001} - 2c_{100}f_{010}) \\ & + 2H(-b_{010}g_{100} - 2f_{100}h_{010} - b_{100}g_{010} + b_{100}h_{001} + b_{001}a_{010} - 2a_{010}f_{010}) \\ & + 2F(-b_{010}c_{100} - b_{100}c_{010} + b_{001}g_{010} + 2b_{001}f_{100} - b_{001}h_{001} - 2f_{010}g_{010} - 4f_{100}f_{010} + 2f_{010}h_{001}) \\ & + 2G(-2c_{100}h_{010} + b_{100}c_{100} + b_{001}g_{100} - a_{010}c_{010} - f^2_{100} - 2f_{100}g_{010} + 2f_{100}h_{001} + 2g_{010}h_{001} - g^2_{010} - h^2_{001} - 2f_{010}g_{100});\end{aligned}$$

$$\alpha_5 = 2L^2 \{A(-f_{100} + g_{010} + h_{001}) + Hb_{001} + Gc_{010}\},$$

$$\beta_5 = 2L^2 \{Aa_{001} + Bb_{001} + Gc_{100} + Fc_{010} + 2Hh_{001}\},$$

$$\gamma_5 = 2L^2 \{Aa_{010} + Hb_{100} + Fb_{001} + Cc_{010} + 2Gg_{010}\},$$

$$\delta_5 = 2L^2 \{Ha_{001} + B(f_{100} - g_{010} + h_{001}) + Fc_{100}\},$$

$$\epsilon_5 = 2L^2 \{Ha_{010} + Ga_{001} + Bb_{100} + Cc_{100} + 2Ff_{100}\},$$

$$\zeta_5 = 2L^2 \{Ga_{010} + Fb_{100} + C(f_{100} + g_{010} - h_{001})\},$$

$$\begin{aligned}\lambda_5 = {} & A(a_{100}f_{100} - a_{100}g_{010} - a_{100}h_{001} - 3a_{010}a_{001}) + H(-a_{100}b_{001} - 4a_{010}h_{001} - a_{001}b_{100} - 2a_{001}h_{010}) \\ & + G(-a_{100}c_{010} - a_{010}c_{100} - 2a_{010}g_{001} - 4a_{001}g_{010}) \\ & + B(-a_{010}b_{001} - b_{100}h_{001} - b_{100}g_{010} + b_{100}f_{100} - 2f_{100}h_{010} + 2g_{010}h_{010} - 2h_{010}h_{001}) \\ & + F(-a_{010}c_{010} - a_{001}b_{001} + b_{100}c_{100} - 2b_{100}g_{001} - 2c_{100}h_{010} + f^2_{100} - 2f_{100}g_{010} - 2f_{100}h_{001} + g^2_{010} - 2g_{010}h_{001} + h^2_{001}) \\ & + C(-a_{001}c_{010} + c_{100}f_{100} - c_{100}g_{010} - c_{100}h_{001} - 2f_{100}g_{001} - 2g_{010}g_{001} + 2g_{001}h_{001}),\end{aligned}$$

$$\begin{aligned}\mu_5 = {} & A(-a_{001}b_{100} - a_{010}h_{001} - a_{010}f_{100} + a_{010}g_{010} - 2g_{010}h_{100} + 2f_{100}h_{100} - 2h_{100}h_{001}) \\ & + H(-a_{001}b_{010} - 4b_{100}h_{001} - a_{010}b_{001} - 2b_{001}h_{100}) + B(b_{010}g_{010} - b_{010}f_{100} - b_{010}h_{001} - 3b_{100}b_{001}) \\ & + F(-b_{010}c_{100} - b_{100}c_{010} - 2b_{100}f_{001} - 4b_{001}f_{100}) \\ & + G(-b_{100}c_{100} - a_{001}b_{001} + a_{010}c_{010} - 2a_{010}f_{001} - 2c_{010}h_{100} + f^2_{100} - 2f_{100}g_{010} - 2f_{100}h_{001} + g^2_{010} - 2g_{010}h_{001} + h^2_{001}) \\ & + C(-b_{001}c_{100} + c_{010}g_{010} - c_{010}f_{100} - c_{010}h_{001} - 2f_{001}g_{010} - 2f_{100}f_{001} + 2f_{001}h_{001}),\end{aligned}$$

$$\begin{aligned}\nu_5 = {} & A(-a_{010}c_{100} + a_{001}h_{001} - a_{001}g_{010} - a_{001}f_{100} - 2g_{100}h_{001} - 2g_{100}g_{010} + 2f_{100}g_{100}) \\ & + H(-a_{010}c_{010} - b_{100}c_{100} + a_{001}b_{001} - 2b_{001}g_{100} - 2a_{001}f_{010} + f^2_{100} - 2f_{100}g_{010} - 2f_{100}h_{001} + g^2_{010} - 2g_{010}h_{001} + h^2_{001}) \\ & + B(-b_{100}c_{010} - b_{001}f_{100} - b_{001}g_{010} + b_{001}h_{001} - 2f_{010}h_{001} + 2f_{010}g_{010} - 2f_{100}f_{010}) \\ & + G(-a_{010}c_{001} - a_{001}c_{010} - 2c_{010}g_{100} - 4c_{100}g_{010}) + F(-b_{100}c_{001} - 4c_{010}f_{100} - b_{001}c_{100} - 2c_{100}f_{010}) \\ & + C(c_{001}h_{001} - c_{001}g_{010} - c_{001}f_{100} - 3c_{100}c_{010});\end{aligned}$$

$$\alpha_6 = 2L^2 \{A(2g_{001} - c_{100}) + H(2f_{001} - c_{010}) + Gc_{001}\},$$

$$\beta_6 = 2L^2 \{H(2g_{001} - c_{100}) + B(2f_{001} - c_{010}) + Fc_{001}\},$$

$$\gamma_6 = 2L^2 \{2Aa_{001} + 2H(f_{100} - g_{010} + h_{001}) + G(2g_{001} + c_{100}) + F(2f_{001} - c_{010}) + Cc_{001}\},$$

$$\epsilon_6 = 4L^2 \{Ha_{001} + B(f_{100} - g_{010} + h_{001}) + Fc_{100}\},$$

$$\zeta_6 = 4L^2 \{Ga_{001} + F(f_{100} - g_{010} + h_{001}) + Cc_{100}\},$$

$$\lambda_6 = A(a_{100}c_{100} - 3a^2_{001} - 2a_{100}g_{001}) + C(c^2_{100} - 4c_{100}g_{001} - a_{001}c_{001})$$
$$+ B(a_{010}c_{010} - 2a_{010}f_{001} - 2f_{100}h_{001} + 2g_{010}h_{001} - 2f_{100}g_{010} - 3h^2_{001} + f^2_{100} + g^2_{010})$$
$$+ F(- a_{010}c_{001} - 2c_{100}h_{001} + 2c_{100}f_{100} - 2c_{100}g_{010} + a_{001}c_{010} - 2a_{001}f_{001} - 4g_{001}h_{001} - 4f_{100}g_{001} + 4g_{010}g_{001})$$
$$+ G(- a_{100}c_{001} - a_{001}c_{100} - 6a_{001}g_{001}) + H(a_{010}c_{100} + a_{100}c_{010} - 6a_{001}h_{001} - 2a_{001}f_{100} + 2a_{001}g_{010} - 2a_{100}f_{001} - 2a_{010}g_{001}),$$

$$\mu_6 = 2A(- 2g_{001}h_{100} + c_{100}h_{100} - a_{001}h_{001} - a_{001}f_{100} + a_{001}g_{010}) + 2B(b_{100}c_{010} - b_{001}h_{001} - b_{001}f_{100} + b_{001}g_{010} - 2b_{100}f_{001})$$
$$+ 2C(- c_{001}f_{100} - 2c_{100}f_{001} + c_{100}c_{010}) + 2G(- c_{001}h_{100} - 2f_{100}g_{001} - c_{100}h_{001} + c_{100}g_{010} + c_{010}a_{001} - 2a_{001}f_{001})$$
$$+ 2F(- b_{100}c_{001} - b_{001}c_{100} + c_{010}h_{001} + 2c_{010}f_{100} - c_{010}g_{010} - 2f_{001}h_{001} - 4f_{100}f_{001} + 2f_{001}g_{010}),$$
$$+ 2H(- 2b_{100}g_{001} + b_{100}c_{100} + c_{010}h_{100} - a_{001}b_{001} - f^2_{100} + 2f_{100}g_{010} - 2f_{100}h_{001} + 2g_{010}h_{001} - g^2_{010} - h^2_{001} - 2f_{001}h_{100})$$

$$\nu_6 = 2A(c_{100}g_{100} - a_{001}c_{100} - 2g_{100}g_{001}) + 2B(- c_{010}h_{001} + c_{010}g_{010} - 2f_{100}f_{001}) - 4Cc_{100}c_{001} - 2G(a_{001}c_{001} + c_{001}g_{100} + 2c_{100}g_{001})$$
$$+ 2F(- c_{001}h_{001} - 2c_{001}f_{100} + c_{001}g_{010} - 2c_{100}f_{001}) + 2H(c_{010}g_{100} - a_{001}c_{010} + c_{100}g_{010} - c_{100}h_{001} - 2f_{001}g_{100} - 2f_{100}g_{001});$$

$$\alpha_7 = 6L^2\{H(2h_{010} - b_{100}) + Bb_{010} + F(2f_{010} - b_{001})\},$$

$$\beta_7 = 6L^2\{A(2h_{010} - b_{100}) + Hb_{010} + G(2f_{010} - b_{001})\},$$

$$\gamma_7 = 6L^2\{G(2h_{010} - b_{100}) + Fb_{010} + C(2f_{010} - b_{001})\},$$

$$\begin{aligned}\lambda_7 = {} & 2A(2a_{010}b_{100} - 4a_{010}h_{010}) + 2B(b_{100}b_{010} - 3b_{010}h_{010}) \\ & + 2C(2b_{001}g_{010} - b_{001}f_{100} + b_{001}h_{001} - 4f_{010}g_{010} + 2f_{100}f_{010} - 2f_{010}h_{001}) \\ & + 2F(-2b_{010}g_{010} + b_{010}f_{100} - b_{010}h_{001} + 2b_{100}f_{010} - b_{100}b_{001} + 3h_{010}b_{001} - 6f_{010}h_{010}) \\ & + 2G(2b_{100}g_{010} - b_{100}f_{100} + b_{100}h_{001} + 2a_{010}b_{001} \\ & \qquad - 4a_{010}f_{010} - 4g_{010}h_{010} + 2f_{100}h_{010} - 2h_{010}h_{001}) \\ & + 2H(-2a_{010}b_{010} + 5b_{100}h_{010} - b^2_{100} - 6h^2_{010}),\end{aligned}$$

$$\begin{aligned}\mu_7 = {} & 2A(b^2_{100} - b_{100}h_{010} - 2h^2_{010}) - 4Bb^2_{010} + 2C(b^2_{001} - b_{001}f_{010} - 2f^2_{010}) \\ & + 2G(2b_{100}b_{001} - b_{100}f_{010} - b_{001}h_{010} - 4f_{010}h_{010}) \\ & + 2F(b_{010}b_{001} - 5b_{010}f_{010}) + 2H(b_{100}b_{010} - 5b_{010}h_{010}),\end{aligned}$$

$$\begin{aligned}\nu_7 = {} & 2C(2b_{001}c_{010} - 4c_{010}f_{010}) + 2B(b_{010}b_{001} - 3b_{010}f_{010}) \\ & + 2A(2b_{100}g_{010} - b_{100}h_{001} + b_{100}f_{100} - 4g_{010}h_{010} + 2h_{010}h_{001} - 2f_{100}h_{010}) \\ & + 2G(2b_{001}g_{010} - b_{001}h_{001} + b_{001}f_{100} + 2b_{100}c_{010} \\ & \qquad - 4c_{010}h_{010} - 4f_{010}g_{010} + 2f_{010}h_{001} - 2f_{100}f_{010}) \\ & + 2F(-2b_{010}c_{010} + 5b_{001}f_{010} - b^2_{001} - 6f^2_{010}) \\ & + 2H(-2b_{010}g_{010} + b_{010}h_{001} - b_{010}f_{100} + 2b_{001}h_{010} - b_{100}b_{001} + 3b_{100}f_{010} - 6f_{010}h_{010});\end{aligned}$$

$$\beta_8 = 4L^2 \{A(-f_{100} + g_{010} + h_{001}) + Hb_{001} + Gc_{010}\},$$

$$\gamma_8 = 2L^2 \{A(2h_{010} - b_{100}) + Hb_{010} + G(2f_{010} - b_{001})\},$$

$$\delta_8 = 4L^2 \{H(-f_{100} + g_{010} + h_{001}) + Bb_{001} + Fc_{010}\},$$

$$\epsilon_8 = 2L^2 \{H(2h_{010} - b_{100}) + 2G(-f_{100} + g_{010} + h_{001}) + Bb_{010} + F(2f_{010} + b_{001}) + 2Cc_{010}\},$$

$$\zeta_8 = 2L^2 \{G(2h_{010} - b_{100}) + Fb_{010} + C(2f_{010} - b_{001})\},$$

$$\begin{aligned}\lambda_8 = {} & 2A(a_{001}b_{100} - a_{010}g_{010} - a_{010}h_{001} + a_{010}f_{100} - 2a_{001}h_{010}) + 2B(-b_{010}h_{001} - 2b_{001}h_{010} + b_{100}b_{001}) \\ & + 2C(-2f_{010}g_{001} + b_{001}g_{001} - c_{010}g_{010} - c_{010}h_{001} + c_{010}f_{100}) \\ & + 2F(-b_{010}g_{001} - 2f_{010}h_{001} - b_{001}g_{010} + b_{001}f_{100} + b_{100}c_{010} - 2c_{010}h_{010}) \\ & + 2G(-2a_{001}f_{010} + a_{001}b_{001} + b_{100}g_{001} - a_{010}c_{010} - f^2_{100} + 2f_{100}g_{010} + 2f_{100}h_{001} - g^2_{010} - 2g_{010}h_{001} - h^2_{001} - 2g_{001}h_{010}) \\ & + 2H(-a_{001}b_{010} - a_{010}b_{001} + b_{100}g_{010} + 2b_{100}h_{001} - b_{100}f_{100} - 2g_{010}h_{010} - 4h_{010}h_{001} + 2f_{100}h_{010}),\end{aligned}$$

$$\begin{aligned}\mu_8 = {} & 2A(-b_{100}g_{010} + b_{100}f_{100} - 2h_{010}h_{001}) - 4Bb_{010}b_{001} + 2C(b_{001}f_{001} - b_{001}c_{010} - 2f_{001}f_{010}) - 2F(b_{010}c_{010} + b_{010}f_{001} + 2b_{001}f_{010}) \\ & + 2G(b_{100}f_{001} - b_{100}c_{010} + b_{001}f_{100} - b_{001}g_{010} - 2f_{001}h_{010} - 2f_{010}h_{001}) + 2H(-b_{010}g_{010} - 2b_{010}h_{001} + b_{010}f_{100} - 2b_{001}h_{010}),\end{aligned}$$

$$\begin{aligned}\nu_8 = {} & A(b_{100}c_{100} - 2c_{100}h_{010} - 2g_{010}h_{001} + 2f_{100}g_{010} - 2f_{100}h_{001} - 3g^2_{010} + f^2_{100} + h^2_{001}) + B(b^2_{001} - 4b_{001}f_{010} - b_{010}c_{010}) \\ & + C(b_{001}c_{001} - 3c^2_{010} - 2c_{001}f_{010}) + F(-b_{010}c_{001} - b_{001}c_{010} - 6c_{010}f_{010}) \\ & + H(-b_{010}c_{100} - 2b_{001}g_{010} + 2b_{001}h_{001} - 2b_{001}f_{100} + b_{100}c_{010} - 2c_{010}h_{010} - 4f_{010}g_{010} - 4f_{010}h_{001} + 4f_{100}f_{010}) \\ & + G(b_{001}c_{100} + b_{100}c_{001} - 6c_{010}g_{010} - 2c_{010}h_{001} + 2c_{010}f_{100} - 2c_{001}h_{010} - 2c_{100}f_{010});\end{aligned}$$

$$\beta_9 = 2L^2 \{A(2g_{001} - c_{100}) + H(2f_{001} - c_{010}) + Gc_{001}\},$$

$$\gamma_9 = 4L^2 \{A(-f_{100} + g_{010} + h_{001}) + Hb_{001} + Gc_{010}\},$$

$$\delta_9 = 2L^2 \{H(2g_{001} - c_{100}) + B(2f_{001} - c_{010}) + Fc_{001}\},$$

$$\epsilon_9 = 2L^2 \{2H(-f_{100} + g_{010} + h_{001}) + G(2g_{001} - c_{100}) + 2Bb_{001} + F(2f_{001} + c_{010}) + Cc_{001}\},$$

$$\zeta_9 = 4L^2 \{G(-f_{100} + g_{010} + h_{001}) + Fb_{001} + Cc_{010}\},$$

$$\begin{aligned}\lambda_9 = {} & 2A(a_{010}c_{100} - a_{001}h_{001} - a_{001}g_{010} + a_{001}f_{100} - 2a_{010}g_{001}) + 2B(-2f_{001}h_{010} + c_{010}h_{010} - b_{001}h_{001} - b_{001}g_{010} + b_{001}f_{100}) \\ & + 2C(-c_{001}g_{010} - 2c_{010}g_{001} + c_{100}c_{010}) + 2F(-c_{001}h_{010} - 2f_{001}g_{010} - c_{010}h_{001} + c_{010}f_{100} + c_{100}b_{001} - 2b_{001}g_{001}) \\ & + 2G(-a_{010}c_{001} - a_{001}c_{010} + c_{100}h_{001} + 2c_{100}g_{010} - c_{100}f_{100} - 2g_{001}h_{001} - 4g_{010}g_{001} + 2f_{100}g_{001}) \\ & + 2H(-2a_{010}f_{001} + a_{010}c_{010} + c_{100}h_{010} - a_{001}b_{001} - f^2_{100} + 2f_{100}h_{001} + 2f_{100}g_{010} - g^2_{010} - 2g_{010}h_{001} - h^2_{001} - 2g_{001}h_{010}),\end{aligned}$$

$$\begin{aligned}\mu_9 = {} & A(b_{100}c_{100} - 2b_{100}g_{001} - 2g_{010}h_{001} + 2f_{100}h_{001} - 2f_{100}g_{010} - 3h^2_{001} + f^2_{100} + g^2_{010}) \\ & + B(b_{010}c_{010} - 3b^2_{001} - 2b_{010}f_{001}) + C(c^2_{010} - 4c_{010}f_{001} - b_{001}c_{001}) \\ & + F(-b_{010}c_{001} - b_{001}c_{010} - 6b_{001}f_{001}) + H(b_{100}c_{010} + b_{010}c_{100} - 6b_{001}h_{001} - 2b_{001}g_{010} + 2b_{001}f_{100} - 2b_{010}g_{001} - 2b_{100}f_{001}) \\ & + G(-b_{100}c_{001} - 2c_{010}h_{001} + 2c_{010}g_{010} - 2c_{010}f_{100} + b_{001}c_{100} - 2b_{001}g_{001} - 4f_{001}h_{001} - 4f_{001}g_{010} + 4f_{100}f_{001}),\end{aligned}$$

$$\begin{aligned}\nu_9 = {} & 2A(-c_{100}h_{001} + c_{100}f_{100} - 2g_{010}g_{001}) + 2B(c_{010}f_{010} - b_{001}c_{010} - 2f_{010}f_{001}) - 4Cc_{010}c_{001} - 2F(b_{001}c_{001} + c_{001}f_{010} + 2c_{010}f_{001}) \\ & + 2G(-c_{001}h_{001} - 2c_{001}g_{010} + c_{001}f_{100} - 2c_{010}g_{001}) + 2H(c_{100}f_{010} - b_{001}c_{100} + c_{010}f_{100} - c_{010}h_{001} - 2f_{010}g_{001} - 2f_{001}g_{010});\end{aligned}$$

$$\alpha_{10} = 6L^2\{G(2g_{001} - c_{100}) + F(2f_{001} - c_{010}) + Cc_{001}\},$$

$$\beta_{10} = 6L^2\{H(2g_{001} - c_{100}) + B(2f_{001} - c_{010}) + Fc_{001}\},$$

$$\gamma_{10} = 6L^2\{A(2g_{001} - c_{100}) + H(2f_{001} - c_{010}) + Gc_{001}\},$$

$$\begin{aligned}\lambda_{10} = {} & 2B(2c_{010}h_{001} - c_{010}f_{100} + c_{010}g_{010} - 4f_{001}h_{001} + 2f_{100}f_{001} - 2f_{001}g_{010}) \\ & + 2A(2a_{001}c_{100} - 4a_{001}g_{001}) + 2C(c_{100}c_{001} - 3c_{001}g_{001}) \\ & + 2F(-2c_{001}h_{001} + c_{001}f_{100} - c_{001}g_{010} + 2c_{100}f_{001} - c_{100}c_{010} + 3g_{001}c_{010} - 6f_{001}g_{001}) \\ & + 2G(-2a_{001}c_{001} + 5c_{100}g_{001} - c^2_{100} - 6g^2_{001}) \\ & + 2H(2c_{100}h_{001} - c_{100}f_{100} + c_{100}g_{010} + 2a_{001}c_{010} \\ & \qquad - 4a_{001}f_{001} - 4g_{001}h_{001} + 2f_{100}g_{001} - 2g_{010}g_{001}),\end{aligned}$$

$$\begin{aligned}\mu_{10} = {} & 2A(2c_{100}h_{001} - c_{100}g_{010} + c_{100}f_{100} - 4g_{001}h_{001} + 2g_{010}g_{001} - 2f_{100}g_{001}) \\ & + 2B(2b_{001}c_{010} - 4b_{001}f_{001}) + 2C(c_{010}c_{001} - 3c_{001}f_{001}) \\ & + 2F(-2b_{001}c_{001} + 5c_{010}f_{001} - c^2_{010} - 6f^2_{001}) \\ & + 2G(-2c_{001}h_{001} + c_{001}g_{010} - c_{001}f_{100} + 2c_{010}g_{001} - c_{100}c_{010} + 3c_{100}f_{001} - 6f_{001}g_{001}) \\ & + 2H(2c_{010}h_{001} - c_{010}g_{010} + c_{010}f_{100} + 2b_{001}c_{100} \\ & \qquad - 4b_{001}g_{001} - 4f_{001}h_{001} + 2f_{001}g_{010} - 2f_{100}f_{001}),\end{aligned}$$

$$\begin{aligned}\nu_{10} = {} & 2A(c^2_{100} - c_{100}g_{001} - 2g^2_{001}) + 2B(c^2_{010} - c_{010}f_{001} - 2f^2_{001}) - 4Cc^2_{001} \\ & + 2F(c_{010}c_{001} - 5c_{001}f_{001}) + 2G(c_{100}c_{001} - 5c_{001}g_{001}) \\ & + 2H(2c_{100}c_{010} - c_{010}g_{001} - c_{100}f_{001} - 4f_{001}g_{001}).\end{aligned}$$

24. Before proceeding to the third set of equations, one substantial simplification is possible. The various quantities that have been obtained are algebraically independent, so far as they occur as solutions of partial differential equations. But there may be intrinsic relations among them owing to the original properties of the quantities which they involve; such intrinsic relations are known to exist among the differential invariants of a surface.

As a matter of fact, *each of the six equations* $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6$ *is equal to zero,* a result established* by CAYLEY in a somewhat different form. The six equations that thus arise, in the case where the independent variables $u$, $v$, $w$ correspond to a triply orthogonal system, are the well known six relations given† by LAMÉ.

As the six quantities $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6$ are permanently zero, we may use this property to simplify the equations in the next set.

The evanescence of these quantities might have been used earlier, in order to modify some of the preceding expressions; but no substantial advantage would have

* 'Coll. Math. Papers,' vol. xii., pp. 12, 13.

† A reference is made by CAYLEY, *loc. cit.*, p. 17.

been derived from their use and there would have been the disadvantage that modification of the formal expressions, while changing their original form without any obvious necessity or obvious benefit, removes that form in which they naturally arise.

25. The third set of equations consists of nine numbers. By modifying them in the same way as the corresponding nine equations in § 10, we obtain similar results. Thus with the old notation for the operator $\Delta_1$ but with the extended significance due to the occurrence of derivatives of $\phi$ of the third order and derivatives of $a, b, c, f, g, h$ of the second order, we find

$$\Delta_1 a'' = 3h'' + Q\Theta_3 - 2R\Theta_4,$$

$$\Delta_1 h'' = 2b'' + P\Theta_3 - 2R\Theta_5,$$

$$\Delta_1 b'' = k'',$$

$$\Delta_1 k'' = 0,$$

$$\Delta_1 g'' = 2f'',$$

$$\Delta_1 f'' = l'',$$

$$\Delta_1 l'' = 0,$$

$$\Delta_1 c'' = m'' - 2P\Theta_6 + Q\Theta_1,$$

$$\Delta_1 m'' = P\Theta_1,$$

$$\Delta_1 n'' = 0\,;$$

$$\Delta_1\Theta_1 = 0,\quad \Delta_1\Theta_2 = 2\Theta_6,\quad \Delta_1\Theta_3 = 0,\quad \Delta_1\Theta_4 = -\Theta_5,\quad \Delta_1\Theta_5 = 0,\quad \Delta_1\Theta_6 = \Theta_1\,;$$

$$\Delta_1 P = 0,\quad \Delta_1 Q = -P,\quad \Delta_1 R = 0.$$

When we insert in these equations the zero values of the quantities $\Theta$, the first ten of them become

$$\Delta_1 a'' = 3h'',\qquad \Delta_1 g'' = 2f'',\qquad \Delta_1 c'' = m'',\qquad \Delta_1 n'' = 0\,;$$

$$\Delta_1 h'' = 2b'',\qquad \Delta_1 f'' = l'',\qquad \Delta_1 m'' = 0,$$

$$_1 b'' = k'',\qquad \Delta_1 l'' = 0,$$

$$\Delta_1 k'' = 0,$$

and so for the equations of the other sets.

Proceeding now exactly as in § 16, we find ultimately that the differential invariants involving the quantities $a, b, c, f, g, h$ and their derivatives, as well as the derivatives of $\phi$, up to the respective specified orders, are the invariants and contravariants of the simultaneous ternary forms

$$\begin{array}{ll} aX^2 + 2gXZ + cZ^2, & \mathbf{a}X^2 + 2\mathbf{g}XZ + \mathbf{c}Z^2, \\ + 2hXY + 2fYZ & + 2\mathbf{h}XY + 2\mathbf{f}YZ \\ + bY^2 & + \mathbf{b}Y^2 \end{array}$$

$$\begin{array}{l} \mathbf{a}''X^3 + 3\mathbf{g}''X^2Z + 3\mathbf{c}''XZ^2 + \mathbf{n}''Z^3, \\ + 3\mathbf{h}''X^2Y + 6\mathbf{f}''XYZ + 3\mathbf{m}''YZ^2 \\ + 3\mathbf{b}''XY^2 + 3\mathbf{l}''Y^2Z \\ + \mathbf{k}''Y^3 \end{array}$$

the contragradient variables being $\phi_{100}$, $\phi_{010}$, $\phi_{001}$. Let any such differential invariant be homogeneous in $\phi_{100}$, $\phi_{010}$, $\phi_{001}$ of degree $p$; in $a, b, c, f, g, h$, of degree $q$; in $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h}$ of degree $r$; in $\mathbf{a}'', \mathbf{b}'', \mathbf{c}'', \mathbf{f}'', \mathbf{g}'', \mathbf{h}'', \mathbf{k}'', \mathbf{l}'', \mathbf{m}'', \mathbf{n}''$ of degree $s$; then the index $\mu$ of the differential invariant is given by

$$3\mu = p + 2q + 8r + 15s.$$

Further, as in § 18, there are five equations out of the set which are satisfied by the differential invariant when it is an invariant of the ternary system, and by the leading coefficient when the differential invariant is a contravariant of the ternary system. These equations are

$$e_1(t) = 0, \quad e_2(t) = 0, \quad e_3(t) = 0, \quad e_4(t) = 0, \quad e_5(t) = 0,$$

where the operators $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ are

$$\begin{aligned} e_1 = {} & 2h\frac{\partial}{\partial a} + b\frac{\partial}{\partial h} + f\frac{\partial}{\partial g} \\ & + 2\mathbf{h}\frac{\partial}{\partial \mathbf{a}} + \mathbf{b}\frac{\partial}{\partial \mathbf{h}} + \mathbf{f}\frac{\partial}{\partial \mathbf{g}} \\ & + 3\mathbf{h}''\frac{\partial}{\partial \mathbf{a}''} + 2\mathbf{b}''\frac{\partial}{\partial \mathbf{h}''} + \mathbf{k}''\frac{\partial}{\partial \mathbf{b}''} + 2\mathbf{f}''\frac{\partial}{\partial \mathbf{g}''} + \mathbf{l}''\frac{\partial}{\partial \mathbf{f}''} + \mathbf{m}''\frac{\partial}{\partial \mathbf{c}''}, \end{aligned}$$

$$\begin{aligned} e_2 = {} & 2g\frac{\partial}{\partial a} + c\frac{\partial}{\partial g} + f\frac{\partial}{\partial h} \\ & + 2\mathbf{g}\frac{\partial}{\partial \mathbf{a}} + \mathbf{c}\frac{\partial}{\partial \mathbf{g}} + \mathbf{f}\frac{\partial}{\partial \mathbf{h}} \\ & + 3\mathbf{g}''\frac{\partial}{\partial \mathbf{a}''} + 2\mathbf{c}''\frac{\partial}{\partial \mathbf{g}''} + \mathbf{n}''\frac{\partial}{\partial \mathbf{c}''} + 2\mathbf{f}''\frac{\partial}{\partial \mathbf{h}''} + \mathbf{m}''\frac{\partial}{\partial \mathbf{f}''} + \mathbf{l}''\frac{\partial}{\partial \mathbf{b}''}, \end{aligned}$$

$$\begin{aligned} e_3 = {} & 2f\frac{\partial}{\partial b} + c\frac{\partial}{\partial f} + g\frac{\partial}{\partial h} \\ & + 2\mathbf{f}\frac{\partial}{\partial \mathbf{b}} + \mathbf{c}\frac{\partial}{\partial \mathbf{f}} + \mathbf{g}\frac{\partial}{\partial \mathbf{h}} \\ & + 3\mathbf{l}''\frac{\partial}{\partial \mathbf{k}''} + 2\mathbf{m}''\frac{\partial}{\partial \mathbf{l}''} + \mathbf{n}''\frac{\partial}{\partial \mathbf{m}''} + 2\mathbf{f}''\frac{\partial}{\partial \mathbf{b}''} + \mathbf{c}''\frac{\partial}{\partial \mathbf{f}''} + \mathbf{g}''\frac{\partial}{\partial \mathbf{h}''}, \end{aligned}$$

$$e_4 = 2f\frac{\partial}{\partial c} + b\frac{\partial}{\partial f} + h\frac{\partial}{\partial g}$$
$$+ 2\mathbf{f}\frac{\partial}{\partial \mathbf{c}} + \mathbf{b}\frac{\partial}{\partial \mathbf{f}} + \mathbf{h}\frac{\partial}{\partial \mathbf{g}}$$
$$+ 3\mathbf{m}''\frac{\partial}{\partial \mathbf{n}''} + 2\mathbf{l}''\frac{\partial}{\partial \mathbf{m}''} + \mathbf{k}''\frac{\partial}{\partial \mathbf{l}''} + 2\mathbf{f}''\frac{\partial}{\partial \mathbf{c}''} + \mathbf{b}''\frac{\partial}{\partial \mathbf{f}''} + \mathbf{h}''\frac{\partial}{\partial \mathbf{g}''},$$

$$e_5 = 2b\frac{\partial}{\partial b} - 2c\frac{\partial}{\partial c} + h\frac{\partial}{\partial h} - g\frac{\partial}{\partial g}$$
$$+ 2\mathbf{b}\frac{\partial}{\partial \mathbf{b}} - 2\mathbf{c}\frac{\partial}{\partial \mathbf{c}} + \mathbf{h}\frac{\partial}{\partial \mathbf{h}} - \mathbf{g}\frac{\partial}{\partial \mathbf{g}}$$
$$+ 3\mathbf{k}''\frac{\partial}{\partial \mathbf{k}''} + \mathbf{l}''\frac{\partial}{\partial \mathbf{l}''} - \mathbf{m}''\frac{\partial}{\partial \mathbf{m}''} - 3\mathbf{n}''\frac{\partial}{\partial \mathbf{n}''} + 2\mathbf{b}''\frac{\partial}{\partial \mathbf{b}''} - 2\mathbf{c}''\frac{\partial}{\partial \mathbf{c}''} + \mathbf{h}''\frac{\partial}{\partial \mathbf{h}''} - \mathbf{g}''\frac{\partial}{\partial \mathbf{g}''}.$$

These equations involve 22 arguments; being themselves 5 in number and a complete set, they possess 17 solutions.

### *Aggregate of Invariants of the Third Order.*

26. The asyzygetic aggregate of concomitants of two ternary quadratics and one ternary cubic has not yet been constructed, so far as I am aware; and it therefore is not possible to select from it an algebraically complete aggregate of invariants and contravariants. But the established knowledge of the asyzygetic system of two ternary quadratics* and of the asyzygetic system of the ternary cubic† is sufficient to permit the construction of this algebraically complete aggregate; it is therefore unnecessary to proceed with the formal solution of the preceding five equations.

The most compendious way of expressing the results is to use the customary symbolical notation. We use the umbral symbols adopted in § 17; and, in addition, in connection with the ternary cubic, we introduce umbral symbols $\alpha_1, \alpha_2, \alpha_3$; $\beta_1, \beta_2, \beta_3$; and so on. Then in the usual notation, we can write

$$(a, b, c, f, g, h \between X, Y, Z)^2 = a_x{}^2 = b_x{}^2 = \ldots,$$
$$(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h} \between X, Y, Z)^2 = a'_x{}^2 = b'_x{}^2 = \ldots,$$
$$(\mathbf{a}'', \mathbf{b}'', \mathbf{c}'', \mathbf{f}'', \mathbf{g}'', \mathbf{h}'', \mathbf{k}'', \mathbf{l}'', \mathbf{m}'', \mathbf{n}'' \between X, Y, Z)^3 = \alpha_x{}^3 = \beta_x{}^3 = \ldots.$$

Then an algebraically complete aggregate of invariants and contravariants of the two quadratics and the cubic is constituted by the following seventeen members:

* See the reference in § 16.

† GORDAN, 'Math. Ann.,' vol. 1 (1869), pp. 90–128; GUNDELFINGER, 'Math. Ann.,' vol. 4 (1871), pp. 144–163; CAYLEY, 'Coll. Math. Papers,' vol. 11, pp. 342–356.

$$I_1 = (abc)^2,$$
$$I_2 = (abu)^2,$$
$$I_3 = (aba')^2,$$
$$I_4 = (aa'u)^2,$$
$$I_5 = (aa'b')^2,$$
$$I_6 = (a'b'u)^2,$$
$$I_7 = (a'b'c')^2,$$
$$I_8 = (\alpha ab)(\beta ab)(\alpha\beta u)^2,$$
$$I_9 = (\alpha aa')(\beta aa')(\alpha\beta u)^2,$$
$$I_{10} = (\alpha a'b')(\beta a'b')(\alpha\beta u)^2,$$
$$I_{11} = (\alpha\beta\gamma)(\alpha\beta u)(\alpha\gamma u)(\beta\gamma u),$$
$$I_{12} = (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\delta)(\beta\gamma\delta),$$
$$I_{13} = (\alpha\beta\gamma)(\alpha\beta a)(\alpha\gamma a)(\beta\gamma u),$$
$$I_{14} = (\alpha\beta\gamma)(\alpha\beta a')(\beta\gamma a')(\beta\gamma u),$$
$$I_{15} = (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\epsilon)(\beta\gamma u)(\delta\epsilon u)^2,$$
$$I_{16} = (\alpha\beta\gamma)(\alpha\beta\delta)(\alpha\gamma\epsilon)(\beta\gamma\zeta)(\delta\epsilon\zeta)^2,$$
$$I_{17} = (\alpha\beta u)^2(\gamma\delta u)^2(\beta\gamma u)(\alpha\delta u).$$

The first seven of these are the set of seven in § 17, which belong to the system of two quadratics. The next three are obtained, by replacing $u_1, u_2, u_3$ by $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ respectively, inserting these changes in $I_2, I_4, I_6$ so as to turn them into operators, and then operating upon the known mixed concomitant $\alpha_x\beta_x(\alpha\beta u)^2$ belonging to the cubic. Of the remainder, $I_{11}, I_{12}, I_{15}, I_{16}, I_{17}$ belong to the system of the cubic; $I_{13}$ and $I_{14}$ are obtained by operating on $I_{11}$ with the symbolical operators

$$\left(a_1\frac{\partial}{\partial u_1} + a_2\frac{\partial}{\partial u_2} + a_3\frac{\partial}{\partial u_3}\right)^2 \quad\text{and}\quad \left(a'_1\frac{\partial}{\partial u_1} + a'_2\frac{\partial}{\partial u_2} + a'_3\frac{\partial}{\partial u_3}\right)^2$$

respectively, or, what is the same thing, by the respective operators

$$a\frac{\partial^2}{\partial u_1^2} + b\frac{\partial^2}{\partial u_2^2} + c\frac{\partial^2}{\partial u_3^2} + 2f\frac{\partial^2}{\partial u_2\,\partial u_3} + 2g\frac{\partial^2}{\partial u_3\,\partial u_1} + 2h\frac{\partial^2}{\partial u_1\,\partial u_2},$$

$$\mathbf{a}\frac{\partial^2}{\partial u_1^2} + \mathbf{b}\frac{\partial^2}{\partial u_2^2} + \mathbf{c}\frac{\partial^2}{\partial u_3^2} + 2\mathbf{f}\frac{\partial^2}{\partial u_2\,\partial u_3} + 2\mathbf{g}\frac{\partial^2}{\partial u_3\,\partial u_1} + 2\mathbf{h}\frac{\partial^2}{\partial u_1\,\partial u_2}.$$

27. All these expressions are given in umbral symbols. Considerable labour might be involved in the process of transforming them so that the changed expressions involve the real symbols; but it is possible to avoid most of this labour by remembering that all the functions are invariants and contravariants of a system of ternary forms, many of which have been calculated and are tabulated in connection with the theory of homogeneous forms. Accordingly, the outline scheme of the real expressions is as follows.

The explicit non-umbral expressions for the first seven of these differential invariants have been given in § 16. The invariant $I_{11}$ is given in explicit form by CAYLEY, who denotes* it by PU. The invariant $I_{15}$ is given in explicit form by CAYLEY, who denotes† it by QU. The invariant $I_{17}$ is given in explicit form by CAYLEY, who denotes‡ it by FU. The invariants $I_{12}$ and $I_{16}$ are given in explicit form by CAYLEY, who denotes§ them by S and T respectively. The invariants $I_{13}$ and $I_{14}$ are obtained by operating on $I_{11}$ with the two operators given at the end of § 26. The actual expression for $I_8$ is obtained by developing the umbral form: it is found to be

$$\begin{aligned} I_8 = \{&A(\mathrm{b}''\mathrm{c}'' - \mathrm{f}''^2) + B(\mathrm{k}''\mathrm{m}'' - \mathrm{l}''^2) + C(\mathrm{l}''\mathrm{n}'' - \mathrm{m}''^2) \\ &+ F(\mathrm{k}''\mathrm{n}'' - \mathrm{l}''\mathrm{m}'') + G(\mathrm{b}''\mathrm{n}'' + \mathrm{c}''\mathrm{l}'' - 2\mathrm{f}''\mathrm{m}'') + H(\mathrm{b}''\mathrm{m}'' + \mathrm{c}''\mathrm{k}'' - 2\mathrm{f}''\mathrm{l}'')\}u_1^2 \\ + \{&A(\mathrm{a}''\mathrm{c}'' - \mathrm{g}''^2) + B(\mathrm{h}''\mathrm{m}'' - \mathrm{f}''^2) + C(\mathrm{g}''\mathrm{n}'' - \mathrm{c}''^2) \\ &+ F(\mathrm{h}''\mathrm{n}'' + \mathrm{g}''\mathrm{m}'' - 2\mathrm{f}''\mathrm{c}'') + G(\mathrm{a}''\mathrm{n}'' - \mathrm{c}''\mathrm{g}'') + H(\mathrm{c}''\mathrm{h}'' + \mathrm{a}''\mathrm{m}'' - 2\mathrm{f}''\mathrm{g}'')\}u_2^2 \\ + \{&A(\mathrm{a}''\mathrm{b}'' - \mathrm{h}''^2) + B(\mathrm{h}''\mathrm{k}'' - \mathrm{b}''^2) + C(\mathrm{g}''\mathrm{l}'' - \mathrm{f}''^2) \\ &+ F(\mathrm{h}''\mathrm{l}'' + \mathrm{g}''\mathrm{k}'' - 2\mathrm{b}''\mathrm{f}'') + G(\mathrm{a}''\mathrm{l}'' + \mathrm{b}''\mathrm{g}'' - 2\mathrm{f}''\mathrm{h}'') + H(\mathrm{a}''\mathrm{k}'' - \mathrm{b}''\mathrm{h}'')\}u_3^2 \\ + 2\{&A(\mathrm{g}''\mathrm{h}'' - \mathrm{a}''\mathrm{f}'') + B(\mathrm{b}''\mathrm{f}'' - \mathrm{h}''\mathrm{l}'') + C(\mathrm{c}''\mathrm{f}'' - \mathrm{g}''\mathrm{m}'') \\ &+ F(\mathrm{f}''^2 + \mathrm{b}''\mathrm{c}'' - \mathrm{h}''\mathrm{m}'' - \mathrm{g}''\mathrm{l}'') + G(\mathrm{c}''\mathrm{h}'' - \mathrm{a}''\mathrm{m}'') + H(\mathrm{b}''\mathrm{g}'' - \mathrm{a}''\mathrm{l}'')\}u_2u_3 \\ + 2\{&A(\mathrm{f}''\mathrm{h}'' - \mathrm{b}''\mathrm{g}'') + B(\mathrm{b}''\mathrm{l}'' - \mathrm{f}''\mathrm{k}'') + C(\mathrm{m}''\mathrm{f}'' - \mathrm{c}''\mathrm{l}'') \\ &+ F(\mathrm{b}''\mathrm{m}'' - \mathrm{c}''\mathrm{k}'') + G(\mathrm{f}''^2 + \mathrm{h}''\mathrm{m}'' - \mathrm{l}''\mathrm{g}'' - \mathrm{b}''\mathrm{c}'') + H(\mathrm{l}''\mathrm{h}'' - \mathrm{g}''\mathrm{k}'')\}u_3u_1 \\ + 2\{&A(\mathrm{f}''\mathrm{g}'' - \mathrm{c}''\mathrm{h}'') + B(\mathrm{f}''\mathrm{l}'' - \mathrm{b}''\mathrm{m}'') + C(\mathrm{c}''\mathrm{m}'' - \mathrm{f}''\mathrm{n}'') \\ &+ F(\mathrm{c}''\mathrm{l}'' - \mathrm{b}''\mathrm{n}'') + G(\mathrm{g}''\mathrm{m}'' - \mathrm{h}''\mathrm{n}'') + H(\mathrm{f}''^2 + \mathrm{l}''\mathrm{g}'' - \mathrm{h}''\mathrm{m}'' - \mathrm{b}''\mathrm{c}'')\}u_1u_2. \end{aligned}$$

* 'Coll. Math. Papers,' vol. 2, p. 326. It should be mentioned that the quantities $a, b, c, f, g, h, i, j, k, l$ in CAYLEY's memoir are to be replaced by a″, k″, n″, l″, c″, h″, m″, g″, b″, f″ respectively, that the quantities $\xi, \eta, \zeta$ in CAYLEY's memoir are to be replaced by $u_1, u_2, u_3$, and that $I_{11}$ contains a numerical factor 6 which can be rejected.

† 'Coll. Math. Papers,' vol. 2, p. 327, with similar modifications and rejection of a numerical factor.

‡ 'Coll. Math. Papers,' vol. 2, p. 328.

§ 'Coll. Math. Papers,' vol. 2, p. 325.

The expression of $I_9$ is obtained by substituting $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ for A, B, C, F, G, H in $I_8$ throughout; and the expression for $I_{10}$ is obtained similarly by substituting $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ for A, B, C, F, G, H in $I_8$ throughout.*

28. All the 17 differential invariants so far given are only relative invariants; it remains to obtain the index of each of them, so as to obtain the absolute invariants. These indices are given by the formula

$$3\mu = p + 2q + 8r + 16s$$

of § 25, and are found to be as follows:—

| | | |
|---|---|---|
| Index | 2, | $I_1, I_2$; |
| . . | 4, | $I_3, I_4$; |
| . . | 6, | $I_5, I_6$; |
| . . | 8, | $I_7$; |
| . . | 12, | $I_8$; |
| . . | 14, | $I_9$; |
| . . | 16, | $I_{10}, I_{11}, I_{13}$; |
| . . | 18, | $I_{14}$; |
| . . | 20, | $I_{12}$; |
| . . | 22, | $I_{17}$; |
| . . | 26, | $I_{15}$; |
| . . | 30, | $I_{16}$. |

Accordingly *there are sixteen algebraically independent absolute differential invariants up to the specified order* in the derivatives of $a$, $b$, $c$, $f$, $g$, $h$ and the derivatives of $\phi$; and an algebraically complete aggregate of these invariants is provided by the set

$$I_2I_1^{-1};\ I_3I_1^{-2},\ I_4I_1^{-2};\ I_5I_1^{-3},\ I_6I_1^{-3};\ I_7I_1^{-4};\ I_8I_1^{-6};\ I_9I_1^{-7};$$

$$I_{10}I_1^{-8},\ I_{11}I_1^{-8},\ I_{13}I_1^{-8};\ I_{14}I_1^{-9};\ I_{12}^{-10};\ I_{17}I_1^{-11};\ I_{15}I_1^{-13};\ I_{16}I_1^{-15}.$$

### *Geometric Significance of the simpler Invariants.*

29. We proceed now to obtain the geometric significance of some of the differential invariants which have been obtained.

The only invariant which is free from derivatives of $a, b, c, f, g, h, \phi, \phi', \phi''$ is the quantity, denoted sometimes by $L^2$ and sometimes by $\Delta_1$. We have

$$\Delta_1 = L^2 = abc + 2fgh - af^2 - bg^2 - ch^2;$$

* A factor 2 needs to be dropped from $I_8$ and $I_{10}$ in passing from the symbolical form to the developed form.

and we have already (§ 3) seen that

$$L = \frac{\partial (x, y, z)}{\partial (u, v, w)}.$$

The invariants, which involve the magnitudes $a, b, c, f, g, h$ but none of their derivatives, and which involve derivatives of $\phi, \phi', \phi''$ of the first order, may be called the differential invariants of the first order.* They are the quantities denoted (§ 19) by $\Theta_1$, $D\Theta_1$, $D^2\Theta_1$, $D'\Theta_1$, $DD'\Theta_1$, $D'^2\Theta_1$, I. In order to give the geometric significance of these invariants, it is necessary to take account of the various directions, through the point $x, y, z$, as determined by the three surfaces $\phi, \phi', \phi''$.

It will be convenient (mainly for the sake of brevity) to adopt an alternative notation† for derivatives of $x, y, z$ with regard to $u, v, w$. We write

$$\frac{\partial x}{\partial u} = x_1, \quad \frac{\partial^2 x}{\partial u^2} = x_{11},$$

$$\frac{\partial x}{\partial v} = x_2, \quad \frac{\partial^2 x}{\partial u\, \partial v} = x_{12}, \quad \frac{\partial^2 x}{\partial v^2} = x_{22},$$

$$\frac{\partial x}{\partial w} = x_3, \quad \frac{\partial^2 x}{\partial u\, \partial w} = x_{13}, \quad \frac{\partial^2 x}{\partial v\, \partial w} = x_{23}, \quad \frac{\partial^2 x}{\partial w^2} = x_{33},$$

and so for derivatives of $y$ and $z$. Also, following CAYLEY, we take

$$\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \zeta_1, \zeta_2, \zeta_3,$$

to be the minors of $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ respectively in

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix}.$$

We then have

$$a = x_1^2 + y_1^2 + z_1^2, \qquad f = x_2 x_3 + y_2 y_3 + z_2 z_3,$$
$$b = x_2^2 + y_2^2 + z_2^2, \qquad g = x_3 x_1 + y_3 y_1 + z_3 z_1,$$
$$c = x_3^2 + y_3^2 + z_3^2, \qquad h = x_1 x_2 + y_1 y_2 + z_1 z_2;$$

$$A = \xi_1^2 + \eta_1^2 + \zeta_1^2, \qquad F = \xi_2 \xi_3 + \eta_2 \eta_3 + \zeta_2 \zeta_3,$$
$$B = \xi_2^2 + \eta_2^2 + \zeta_2^2, \qquad G = \xi_3 \xi_1 + \eta_3 \eta_1 + \zeta_3 \zeta_1,$$
$$C = \xi_3^2 + \eta_3^2 + \zeta_3^2, \qquad H = \xi_1 \xi_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2.$$

Let $dn$ denote an element of distance through the point, in a direction normal to the surface $\phi$ = constant, and let

$$\phi(u, v, w) = \Phi(x, y, z).$$

* The magnitudes $a, b, c, f, g, h$, by their definition, involve derivatives of $x, y, z$ of the first order.

† This is CAYLEY's notation; see note to § 1.

Then
$$\phi_{100} = \Phi_x x_1 + \Phi_y y_1 + \Phi_z z_1,$$
$$\phi_{010} = \Phi_x x_2 + \Phi_y y_2 + \Phi_z z_2,$$
$$\phi_{001} = \Phi_x x_3 + \Phi_y y_3 + \Phi_z z_3.$$

Now
$$\begin{aligned}\Theta &= (\mathrm{A, B, C, F, G, H})(\phi_{100}, \phi_{010}, \phi_{001})^2 \\ &= (\phi_{100}\xi_1 + \phi_{010}\xi_2 + \phi_{001}\xi_3)^2 \\ &\quad + (\phi_{100}\eta_1 + \phi_{010}\eta_2 - \phi_{001}\eta_3)^2 \\ &\quad + (\phi_{100}\zeta_1 + \phi_{010}\zeta_2 + \phi_{001}\zeta_3)^2,\end{aligned}$$

on substituting for A, B, C, F, G, H; when further substitution for $\phi_{100}$, $\phi_{010}$, $\phi_{001}$ from above takes place, we have
$$\Theta_1 = (\Phi_x^2 + \Phi_y^2 + \Phi_z^2)\,\mathrm{L}^2.$$

Now
$$\frac{\dfrac{dx}{dn}}{\Phi_x} = \frac{\dfrac{dy}{dn}}{\Phi_y} = \frac{\dfrac{dz}{dn}}{\Phi_z} = \frac{1}{(\Phi_x^2 + \Phi_y^2 + \Phi_z^2)^{\frac{1}{2}}},$$

so that
$$\begin{aligned}\frac{d\phi}{dn} = \frac{d\Phi}{dn} &= \Phi_x \frac{dx}{dn} + \Phi_y \frac{dy}{dn} + \Phi_z \frac{dz}{dn} \\ &= (\Phi_x^2 + \Phi_y^2 + \Phi_z^2)^{\frac{1}{2}};\end{aligned}$$

and therefore
$$\frac{\Theta_1}{\mathrm{L}^2} = \left(\frac{d\phi}{dn}\right)^2.$$

30. Let $dn'$ denote an element of distance through the point in a direction normal to the surface $\phi' =$ constant; and similarly let $dn''$ denote an element of distance through the point in a direction normal to the surface $\phi'' =$ constant. Also let
$$\phi'(u, v, w) = \Phi'(x, y, z), \quad \phi''(u, v, w) = \Phi''(x, y, z).$$

Then, as
$$2\mathrm{D} = \phi'_{100}\frac{\partial}{\partial\phi_{100}} + \phi'_{010}\frac{\partial}{\partial\phi_{010}} + \phi'_{001}\frac{\partial}{\partial\phi_{001}},$$
$$2\mathrm{D}' = \phi''_{100}\frac{\partial}{\partial\phi_{100}} + \phi''_{010}\frac{\partial}{\partial\phi_{010}} + \phi''_{001}\frac{\partial}{\partial\phi_{001}},$$

we have
$$\begin{aligned}\mathrm{D}\Theta_1 &= (\phi_{100}\xi_1 + \phi_{010}\xi_2 + \phi_{001}\xi_3)(\phi'_{100}\xi_1 + \phi'_{010}\xi_2 + \phi'_{001}\xi_3) \\ &\quad + (\phi_{100}\eta_1 + \phi_{010}\eta_2 + \phi_{001}\eta_3)(\phi'_{100}\eta_1 + \phi'_{010}\eta_2 + \phi'_{001}\eta_3) \\ &\quad + (\phi_{100}\zeta_1 + \phi_{010}\zeta_2 + \phi_{001}\zeta_3)(\phi'_{100}\zeta_1 + \phi'_{010}\zeta_2 + \phi'_{001}\zeta_3) \\ &= \mathrm{L}^2(\Phi_x\Phi'_x + \Phi_y\Phi'_y + \Phi_z\Phi'_z) \\ &= \mathrm{L}^2\frac{d\phi}{dn}\frac{d\phi'}{dn'}\left(\frac{dx}{dn}\frac{dx'}{dn'} + \frac{dy}{dn}\frac{dy'}{dn'} + \frac{dz}{dn}\frac{dz'}{dn'}\right) \\ &= \mathrm{L}^2\frac{d\phi}{dn}\frac{d\phi'}{dn'}\cos\Omega'',\end{aligned}$$

where $\Omega''$ is the angle at which the surfaces intersect at the point, that is, the angle between the normals to the surfaces. Thus

$$\frac{D\Theta_1}{L^2} = \frac{d\phi}{dn}\frac{d\phi'}{dn'}\cos\Omega''.$$

Further, let $\Omega$ be the angle at which the two surfaces $\phi' =$ constant, $\phi'' =$ constant, intersect; and let $\Omega'$ be the angle at which the two surfaces $\phi =$ constant, $\phi'' =$ constant intersect. Then we have similarly

$$\frac{D^2\Theta_1}{L^2} = \left(\frac{d\phi'}{dn'}\right)^2, \qquad \frac{D'\Theta_1}{L^2} = \frac{d\phi}{dn}\frac{d\phi''}{dn''}\cos\Omega',$$

$$\frac{DD'\Theta_1}{L^2} = \frac{d\phi'}{dn'}\frac{d\phi''}{dn''}\cos\Omega, \qquad \frac{D'^2\Theta_1}{L^2} = \left(\frac{d\phi''}{dn''}\right)^2.$$

31. Again, let $ds$ denote an element of arc through the point along the curve of intersection of the surfaces $\phi' =$ constant, $\phi'' =$ constant; let $ds'$ be a similar element along the intersection of $\phi =$ constant, $\phi'' =$ constant; and let $ds''$ be a similar element along the intersection of $\phi =$ constant, $\phi' =$ constant. Then

$$\phi'_{100}\frac{du}{ds} + \phi'_{010}\frac{dv}{ds} + \phi'_{001}\frac{dw}{ds} = 0,$$

$$\phi''_{100}\frac{du}{ds} + \phi''_{010}\frac{dv}{ds} + \phi''_{001}\frac{dw}{ds} = 0;$$

and therefore

$$\frac{1}{\theta_1}\frac{du}{ds} = \frac{1}{\theta_2}\frac{dv}{ds} = \frac{1}{\theta_3}\frac{dw}{ds} = \mu,$$

say, where

$$\theta_1, \theta_2, \theta_3 = \left\| \begin{matrix} \phi'_{100}, & \phi'_{010}, & \phi'_{001} \\ \phi''_{100}, & \phi''_{010}, & \phi''_{001} \end{matrix} \right\|.$$

Now

$$(a, b, c, f, g, h \between du, dv, dw)^2 = ds^2,$$

so that

$$\frac{1}{\mu^2} = (a, b, c, f, g, h \between \theta_1, \theta_2, \theta_3)^2$$
$$= w_2,$$

say. Thus

$$\frac{1}{\theta_1}\frac{du}{ds} = \frac{1}{\theta_2}\frac{dv}{ds} = \frac{1}{\theta_3}\frac{dw}{ds} = \frac{1}{\sqrt{w_2}}.$$

Again, we have

$$w_2 = (a, b, c, f, g, h \between \theta_1, \theta_2, \theta_3)^2$$
$$= (x_1\theta_1 + x_2\theta_2 + x_3\theta_3)^2 + (y_1\theta_1 + y_2\theta_2 + y_3\theta_3)^2 + (z_1\theta_1 + z_2\theta_2 + z_3\theta_3)^2.$$

But

$$x_1\theta_1+x_2\theta_2+x_3\theta_3=\begin{vmatrix} x_1, & x_2, & x_3 \\ \phi'_{100}, & \phi'_{010}, & \phi'_{001} \\ \phi''_{100}, & \phi''_{010}, & \phi''_{001} \end{vmatrix}$$

$$=\begin{vmatrix} x_1, & x_2, & x_3 \\ \Phi'_x x_1+\Phi'_y y_1+\Phi'_z z_1, & \Phi'_x x_2+\Phi'_y y_2+\Phi'_z z_2, & \Phi'_x x_3+\Phi'_y y_3+\Phi'_z z_3 \\ \Phi''_x x_1+\Phi''_y y_1+\Phi''_z z_1, & \Phi''_x x_2+\Phi''_y y_2+\Phi''_z z_2, & \Phi''_x x_3+\Phi''_y y_3+\Phi''_z z_3 \end{vmatrix}$$

$$=(\Phi'_y\Phi''_z-\Phi'_z\Phi''_y)\,\mathrm{L},$$

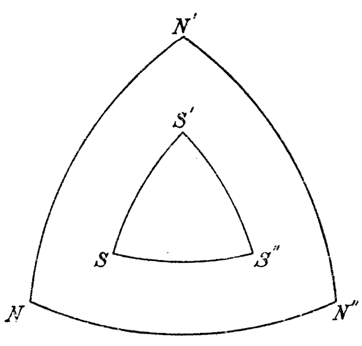
and similarly for $y_1\theta_1+y_2\theta_2+y_3\theta_3$, $z_1\theta_1+z_2\theta_2+z_3\theta_3$; hence

$$w_2=\mathrm{L}^2\{(\Phi'_y\Phi''_z-\Phi'_z\Phi''_y)^2+(\Phi'_z\Phi''_x-\Phi'_x\Phi''_z)^2+(\Phi'_x\Phi''_y-\Phi'_y\Phi''_x)^2\}$$
$$=\mathrm{L}^2\{(\Phi'^2_x+\Phi'^2_y+\Phi'^2_z)(\Phi''^2_x+\Phi''^2_y+\Phi''^2_z)-(\Phi'_x\Phi''_x+\Phi'_y\Phi''_y+\Phi'_z\Phi''_z)^2\}$$
$$=\mathrm{L}^2\left(\frac{d\phi'}{dn'}\right)^2\left(\frac{d\phi''}{dn''}\right)^2\sin^2\Omega.$$

This result is useful for the identification of the invariant I. We have

$$\mathrm{I}=\begin{vmatrix} \phi_{100}, & \phi_{010}, & \phi_{001} \\ \phi'_{100}, & \phi'_{010}, & \phi'_{001} \\ \phi''_{100}, & \phi''_{010}, & \phi''_{001} \end{vmatrix}$$
$$=\theta_1\phi_{100}+\theta_2\phi_{010}+\theta_3\phi_{001}$$
$$=\sqrt{w_2}\left(\phi_{100}\frac{du}{ds}+\phi_{010}\frac{dv}{ds}+\phi_{001}\frac{dw}{ds}\right)$$
$$=\sqrt{w_2}\,\frac{d\phi}{ds}$$
$$=\mathrm{L}\,\frac{d\phi}{ds}\,\frac{d\phi'}{dn'}\,\frac{d\phi''}{dn''}\sin\Omega.$$

This expression can be modified so as to become skew symmetric in the three surfaces. Take a sphere, having its radius unity and its centre at the point; and let the directions $ds$, $ds'$, $ds''$, $dn$, $dn'$, $dn''$ cut the surface at S, S′, S″, N, N′, N″. Then SS′S″ is the polar triangle of NN′N″; and N′N″ $=\Omega$, N″N $=\Omega'$, NN′ $=\Omega''$. By a known property of such triangles, we have




$$\cos \mathrm{NS}\sin\Omega=\cos \mathrm{N'S'}\sin\Omega'=\cos \mathrm{N''S''}\sin\Omega''$$
$$=(1-\cos^2\Omega-\cos^2\Omega'-\cos^2\Omega''+2\cos\Omega\cos\Omega'\cos\Omega'')^{\frac{1}{2}}$$
$$=\nabla,$$

say. Now

$$dn = \cos \mathrm{SN} \,.\, ds,$$

so that

$$\frac{d\phi}{ds} = \frac{d\phi}{dn} \cos \mathrm{SN}.$$

Hence

$$\mathrm{I} = \mathrm{L} \frac{d\phi}{dn} \frac{d\phi'}{dn'} \frac{d\phi''}{dn''} \cos \mathrm{SN} \sin \Omega$$

$$= \mathrm{L}\nabla \frac{d\phi}{dn} \frac{d\phi'}{dn'} \frac{d\phi''}{dn''},$$

the skew symmetric expression required.

It is easy to verify that the values obtained for the invariants satisfy the relation in § 19.

*Significance of the Invariants of the Second Order.*

32. Passing now to the differential invariants of the second order associated with a single surface, we shall adopt another method of identifying them geometrically. The functions are invariantive through all changes of the independent variables and therefore possess the value given by any particular selection of variables. Now one simple transformation of the variables is that which makes $x$, $y$, $z$ respectively equal to the independent variables, so that we take

$$x = u, \quad y = v, \quad z = w.$$

With these values, we have

$$a = 1, \quad b = 1, \quad c = 1, \quad f = 0, \quad g = 0, \quad h = 0;$$
$$\mathrm{A} = 1, \quad \mathrm{B} = 1, \quad \mathrm{C} = 1, \quad \mathrm{F} = 0, \quad \mathrm{G} = 0, \quad \mathrm{H} = 0;$$
$$\mathrm{L}^2 = 1,$$

and

$$\phi_{lmn} = \frac{\partial^{l+m+n}\phi}{\partial x^l \, \partial y^m \, \partial z^n}.$$

Also

$$\mathfrak{a} = 2\phi_{200} = 2\phi_{xx}, \quad \mathfrak{f} = 2\phi_{011} = 2\phi_{yz},$$
$$\mathfrak{b} = 2\phi_{020} = 2\phi_{yy}, \quad \mathfrak{g} = 2\phi_{101} = 2\phi_{xz},$$
$$\mathfrak{c} = 2\phi_{002} = 2\phi_{zz}, \quad \mathfrak{h} = 2\phi_{110} = 2\phi_{xy}.$$

Hence

$$\frac{\Theta_1}{\Delta_1} = \phi_x^2 + \phi_y^2 + \phi_z^2,$$

$$\tfrac{1}{2}\frac{\Theta_{12}}{\Delta_1^2} = (\phi_{yy} + \phi_{zz})\,\phi_x^2 + (\phi_{zz} + \phi_{xx})\,\phi_y^2 + (\phi_{xx} + \phi_{yy})\,\phi_z^2$$
$$- 2\phi_x\phi_y\phi_{xy} - 2\phi_x\phi_z\phi_{xz} - 2\phi_y\phi_z\phi_{yz},$$

$$\tfrac{1}{2}\frac{\Delta_{12}}{\Delta_1^2} = \phi_{xx} + \phi_{yy} + \phi_{zz},$$

$$\frac{1}{4}\frac{\Theta_2}{\Delta_1^3} = -\begin{vmatrix} \phi_{xx}, & \phi_{xy}, & \phi_{xz}, & \phi_x \\ \phi_{xy}, & \phi_{yy}, & \phi_{zy}, & \phi_y \\ \phi_{xz}, & \phi_{yz}, & \phi_{zz}, & \phi_z \\ \phi_x, & \phi_y, & \phi_z, & 0 \end{vmatrix}$$

$$= (\phi_{yy}\phi_{zz} - \phi_{yz}^2)\phi_x^2 + \ldots$$

$$\frac{1}{4}\frac{\Delta_{21}}{\Delta_1^3} = \phi_{yy}\phi_{zz} - \phi_{yz}^2 + \phi_{zz}\phi_{xx} - \phi_{zx}^2 + \phi_{xx}\phi_{yy} - \phi_{xy}^2,$$

$$\frac{1}{8}\frac{\Delta_2}{\Delta_1^4} = \begin{vmatrix} \phi_{xx}, & \phi_{xy}, & \phi_{xz} \\ \phi_{xy}, & \phi_{yy}, & \phi_{yz} \\ \phi_{xz}, & \phi_{yz}, & \phi_{zz} \end{vmatrix}.$$

As usual, let $R_1$ and $R_2$ be the principal radii of curvature of the surface $\phi =$ constant; and let

$$\sigma = \frac{1}{R}(\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{1}{2}},$$

for the radii $R_1 = R$, and $R_2 = R$. Then* the two values of $\sigma$ are given by the equation

$$\begin{vmatrix} \phi_{xx} - \sigma, & \phi_{xy}, & \phi_{xz}, & \phi_x \\ \phi_{xy}, & \phi_{yy} - \sigma, & \phi_{yz}, & \phi_y \\ \phi_{xz}, & \phi_{yz}, & \phi_{zz} - \sigma, & \phi_z \\ \phi_x, & \phi_y, & \phi_z, & 0 \end{vmatrix} = 0;$$

and therefore

$$\sigma_1\sigma_2 = \frac{(\phi_{yy}\phi_{zz} - \phi_{yz}^2)\phi_x^2 + \ldots}{\phi_x^2 + \phi_y^2 + \phi_z^2}$$

$$\sigma_1 + \sigma_2 = \frac{(\phi_{yy} + \phi_{zz})\phi_x^2 + \ldots \quad - 2\phi_x\phi_y\phi_{xy} - \ldots}{\phi_x^2 + \phi_y^2 + \phi_z^2}.$$

Consequently

$$\frac{\Theta_2}{4\Theta_1\Delta_1^2} = \sigma_1\sigma_2 = \frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{\rho_1\rho_2}$$

$$= \frac{\Theta_1}{\Delta_1}\frac{1}{R_1R_2};$$

and therefore

$$\frac{\Theta_2}{\Delta_1^3} = 4\frac{\Theta_1^2}{\Delta_1^2}\frac{1}{R_1R_2}$$

$$= \frac{4}{R_1R_2}\left(\frac{d\phi}{dn}\right)^4.$$

* FROST, 'Solid Geometry,' p. 293





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Similarly

$$\frac{\Theta_{12}}{2\Theta_1\Delta_1} = \sigma_1 + \sigma_2$$

$$= \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{\Theta_1}{\Delta_1}\right)^{\frac{1}{2}};$$

and therefore

$$\frac{\Theta_{12}}{\Delta_1^2} = 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{\Theta_1}{\Delta_1}\right)^{\frac{3}{2}}$$

$$= 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{d\phi}{dn}\right)^3.$$

Again, we have

$$\frac{d}{dn} = \frac{\phi_x \frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y} + \phi_z \frac{\partial}{\partial z}}{(\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{1}{2}}},$$

so that

$$\frac{d\phi}{dn} = (\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{1}{2}},$$

and therefore

$$\frac{d^2\phi}{dn^2} = \frac{\phi_{xx}\phi_x^2 + 2\phi_{xy}\phi_x\phi_y + 2\phi_{xz}\phi_x\phi_z + \phi_{yy}\phi_y^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z^2}{\phi_x^2 + \phi_y^2 + \phi_z^2}.$$

Hence

$$\frac{\Theta_1}{\Delta_1}\frac{d^2\phi}{dn^2} = \tfrac{1}{2}\frac{\Delta_{12}}{\Delta_1^2}\frac{\Theta_1}{\Delta_1} - \tfrac{1}{2}\frac{\Theta_{12}}{\Delta_1^2},$$

that is,

$$\left(\frac{d\phi}{dn}\right)^2\frac{d^2\phi}{dn^2} = \tfrac{1}{2}\frac{\Delta_{12}}{\Delta_1^2}\left(\frac{d\phi}{dn}\right)^2 - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{d\phi}{dn}\right)^3,$$

and therefore

$$\frac{\Delta_{12}}{\Delta_1^2} = 2\frac{d^2\phi}{dn^2} + 2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{d\phi}{dn}.$$

33. Let U, V, W denote $\phi_{100}, \phi_{010}, \phi_{001}$ respectively; then the directions of the lines of curvature, through the point $x, y, z$ on the surface $\phi =$ constant, are given by the equations

$$\begin{vmatrix} dU, & U, & dx \\ dV, & V, & dy \\ dW, & W, & dz \end{vmatrix} = 0,$$

$$U\,dx + V\,dy + W\,dz = 0.$$

Let $l, m, n$ denote the direction-cosines of either line; then

$$\begin{vmatrix} \mathbf{a}l + \mathbf{h}m + \mathbf{g}n, & U, & l \\ \mathbf{h}l + \mathbf{b}m + \mathbf{f}n, & V, & m \\ \mathbf{g}l + \mathbf{f}m + \mathbf{c}n, & W, & n \end{vmatrix} = 0,$$

$$Ul + Vm + Wn = 0.$$

The former of these can be replaced by the three equations

$$\left.\begin{aligned} \mathrm{a}l + \mathrm{h}m + \mathrm{g}n &= \rho \mathrm{U} + \sigma l \\ \mathrm{h}l + \mathrm{b}m + \mathrm{f}n &= \rho \mathrm{V} + \sigma m \\ \mathrm{g}l + \mathrm{f}m + \mathrm{c}n &= \rho \mathrm{W} + \sigma n \end{aligned}\right\} ;$$

and we also have

$$l^2 + m^2 + n^2 = 1.$$

Let $ds$ be an element of arc of a line of curvature; then

$$\begin{aligned} \frac{d}{ds}\left(\frac{d\phi}{dn}\right) &= \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z}\right) \frac{\partial \phi}{\partial n} \\ &= \left(l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z}\right) (\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{1}{2}} \\ &= \frac{l\,(\mathrm{Ua} + \mathrm{Vh} + \mathrm{Wg}) + m\,(\mathrm{Uh} + \mathrm{Vb} + \mathrm{Wf}) + n\,(\mathrm{Ug} + \mathrm{Vf} + \mathrm{Wc})}{2\,(\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2)^{\frac{1}{2}}} . \end{aligned}$$

Multiplying the foregoing three equations by U, V, W and adding, we have

$$\rho\,(\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2) = 2\,(\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2)^{\frac{1}{2}} \frac{d}{ds}\left(\frac{d\phi}{dn}\right),$$

that is,

$$\rho \frac{d\phi}{dn} = 2 \frac{d}{ds}\left(\frac{d\phi}{dn}\right) = 2\mathrm{S},$$

say. Multiplying the same three equations by $l$, $m$, $n$ and adding, we have

$$\begin{aligned} \sigma &= \mathrm{a}l^2 + 2\mathrm{h}lm + 2\mathrm{g}ln + \mathrm{b}m^2 + 2\mathrm{f}mn + \mathrm{c}n^2 \\ &= \frac{2}{\mathrm{R}} \frac{\partial \phi}{\partial n}, \end{aligned}$$

where R is the radius of curvature of the principal normal section at the point.* Thus $\rho$ and $\sigma$ can be regarded as known.

Again, solving the three equations for $l$, $m$, $n$, we obtain three results of the form

$$\begin{vmatrix} \mathrm{a} - \sigma, & \mathrm{h}\ , & \mathrm{g} \\ \mathrm{h}\ , & \mathrm{b} - \sigma, & \mathrm{f} \\ \mathrm{g}\ , & \mathrm{f}\ , & \mathrm{c} - \sigma \end{vmatrix} l = \rho \begin{vmatrix} \mathrm{U}, & \mathrm{h}\ , & \mathrm{g} \\ \mathrm{V}, & \mathrm{b} - \sigma, & \mathrm{f} \\ \mathrm{W}, & \mathrm{f}\ , & \mathrm{c} - \sigma \end{vmatrix} .$$

When these are substituted in $l^2 + m^2 + n^2 = 1$, we have

$$\begin{vmatrix} \mathrm{a} - \sigma, & \mathrm{h}\ , & \mathrm{g} \\ \mathrm{h}\ , & \mathrm{b} - \sigma, & \mathrm{f} \\ \mathrm{g}\ , & \mathrm{f}\ , & \mathrm{c} - \sigma \end{vmatrix}^2 = \rho^2 \Phi,$$

* FROST, 'Solid Geometry,' p. 290.

where, after some reduction, $\Phi$ is found to be given by

$$\Phi = \sigma^4 \frac{\Theta_1}{\Delta_1} - 2\sigma^3 \frac{\Theta_{12}}{\Delta_1^{\ 2}} + \sigma^2 \left( \frac{\Delta_{12}\Theta_{12}}{\Delta_1^{\ 4}} - \frac{\Delta_{21}\Theta_1}{\Delta_1^{\ 4}} + 3\frac{\Theta_2}{\Delta_1^{\ 3}} \right)$$

$$- 2\sigma \frac{\Theta_2\Delta_{12} - \Delta_2\Theta_1}{\Delta_1^{\ 5}} + \frac{\Theta_2\Delta_{21} - \Delta_2\Theta_{12}}{\Delta_1^{\ 6}}.$$

Moreover

$$\begin{vmatrix} a - \sigma, & h \ , & g \\ h \ , & b - \sigma, & f \\ g \ , & f \ , & c - \sigma \end{vmatrix} = \frac{\Delta_2}{\Delta_1^{\ 4}} - \frac{\Delta_{21}}{\Delta_1^{\ 3}}\sigma + \frac{\Delta_{12}}{\Delta_1^{\ 2}}\sigma^2 - \sigma^3 ;$$

consequently

$$\frac{1}{\rho^2}\left( \frac{\Delta_2}{\Delta_1^{\ 4}} - \frac{\Delta_{21}}{\Delta_1^{\ 3}}\sigma + \frac{\Delta_{12}}{\Delta_1^{\ 2}}\sigma^2 - \sigma^3 \right)^2$$

$$= \sigma^4 \frac{\Theta_1}{\Delta_1} - 2\sigma^3 \frac{\Theta_{12}}{\Delta_1^{\ 2}} + \frac{\sigma^2}{\Delta_1^{\ 4}} (\Delta_{12}\Theta_{12} - \Delta_{21}\Theta_1 + 3\Delta_1\Theta_2)$$

$$- \frac{2\sigma}{\Delta_1^{\ 5}} (\Theta_2\Delta_{12} - \Delta_2\Theta_1) + \frac{1}{\Delta_1^{\ 6}} (\Theta_2\Delta_{21} - \Delta_2\Theta_{12}).$$

When $\sigma = \sigma_1$, we may take

$$\rho = \rho_1 = \frac{2S_1}{\frac{d\phi}{dn}} = \frac{2}{\frac{d\phi}{dn}} \frac{d}{ds_1}\left(\frac{d\phi}{dn}\right);$$

when $\sigma = \sigma_2$, we may take

$$\rho = \rho_2 = \frac{2S_2}{\frac{d\phi}{dn}} = \frac{2}{\frac{d\phi}{dn}} \frac{d}{ds_2}\left(\frac{d\phi}{dn}\right).$$

The two equations which thus arise by taking $\sigma = \sigma_1$, and $\sigma = \sigma_2$, when combined symmetrically and rationally, may be regarded as two equations defining $\Delta_{21}$ and $\Delta_2$ algebraically in terms of the quantities $\frac{d}{ds_1}\left(\frac{d\phi}{dn}\right)$ and $\frac{d}{ds_2}\left(\frac{d\phi}{dn}\right)$.

34. The geometric significance of most, if not all, the differential invariants of the second order, which occur (§ 21) in the algebraic aggregate when two surfaces are given, can be obtained by noting the properties for the second surface corresponding to those discussed for the first surface; they will not be considered further in this memoir.

Nor is it my intention in this place to discuss the geometric significance of the sixteen members of the algebraic aggregate of differential invariants up to the third order inclusive which occur (§ 28) when there is only a single surface. But one remark may be made. Out of the aggregate of sixteen, six already have been identified; they are invariants of order less than three, so that there remains the identification of the ten invariants which actually involve quantities of the third order. Let H and K denote the mean curvature and the specific curvature respectively of the surface $\phi =$ constant, so that

$$H = \frac{1}{R_1} + \frac{1}{R_2}, \qquad K = \frac{1}{R_1 R_2}.$$

Then the geometrical magnitudes available for the identification of the invariants are

$$\frac{dH}{dn}, \qquad \frac{dH}{ds_1}, \qquad \frac{dH}{ds_2},$$

$$\frac{dK}{dn}, \qquad \frac{dK}{ds_1}, \qquad \frac{dK}{ds_2},$$

$$\frac{d^3\phi}{dn^3}, \qquad \frac{d}{ds_1}\left(\frac{d^2\phi}{dn^2}\right), \qquad \frac{d}{ds_2}\left(\frac{d^2\phi}{dn^2}\right),$$

$$\frac{d^2}{ds_1^2}\left(\frac{d\phi}{dn}\right), \qquad \frac{d^2}{ds_1\, ds_2}\left(\frac{d\phi}{dn}\right), \qquad \frac{d^2}{ds_2^2}\left(\frac{d\phi}{dn}\right),$$

being twelve in number. Accordingly, it is to be expected that, among these quantities and the quantities already used, which are

$$H, \quad K, \quad \frac{d\phi}{dn}, \quad \frac{d^2\phi}{dn^2}, \quad \frac{d}{ds_1}\left(\frac{d\phi}{dn}\right), \quad \frac{d}{ds_2}\left(\frac{d\phi}{dn}\right),$$

two algebraical relations exist.